

Math 846

Lecture 90

$\mathbb{F}$ arb $0 \neq q \in \mathbb{F}$ q not a root of 1.

We return our attention to the q -Onsager algebra $\mathcal{O}_q$.

Our next goal is to describe two automorphisms of $\mathcal{O}_q$, called the Lusztig automorphisms.

To motivate these automorphisms, we recall the $\mathrm{PSL}_2(\mathbb{Z})$ action on Δ_q as a group of automorphisms

u	A	B	C	α	β	γ
$p(u)$	B	C	A	β	γ	α
$s(u)$	B	A	$C + \frac{AB-BA}{q-q^{-1}}$	β	α	γ

The $\mathrm{PSL}_2(\mathbb{Z})$ generators p, s each permute α, β, γ . This gives a group hom from $\mathrm{PSL}_2(\mathbb{Z})$ into the symmetric group S_3 . Let Γ denote the kernel of this hom. Thus Γ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and the quotient group $\mathrm{PSL}_2(\mathbb{Z})/\Gamma$ is isomorphic to S_3 .

The cosets of $\mathbb{P}$ in $PSL_2(\mathbb{Z})$ are

$$\mathbb{P} \quad p\mathbb{P} \quad p^2\mathbb{P} \quad s\mathbb{P} \quad ps\mathbb{P} \quad p^2s\mathbb{P}.$$

Define

$$T_0 = (p^2s)^2 = (sp)^{-2}, \quad T_1 = (sp^2)^2 = (ps)^{-2}$$

Using the previous table, we find

$$T_0, T_1 \in \mathbb{P}.$$

We remark that $\mathbb{P}$ is freely generated by $T_0^{\pm 1}, T_1^{\pm 1}$.

We now compute the actions of $T_0^{\pm 1}, T_1^{\pm 1}$ on A, B .

LEM 64 We have

$$T_0(A) = A, \quad T_0(B) = B + \frac{[A, [A, B]_q]}{(1-q^{-1})(q^2 - q^{-2})}$$

$$T_1(B) = B, \quad T_1(A) = A + \frac{[B, [B, A]_q]}{(1-q^{-1})(q^2 - q^{-2})}$$

$$T_0^{-1}(A) = A, \quad T_0^{-1}(B) = B + \frac{[A, [A, B]_{q^{-1}}]}{(1-q)(q^2 - q^{-2})}$$

$$T_1^{-1}(B) = B, \quad T_1^{-1}(A) = A + \frac{[B, [B, A]_{q^{-1}}]}{(1-q)(q^2 - q^{-2})}$$

pf Use the previous table.

□

The following result is motivated by Lem 64.

Thm 65. The q -Onsager algebra has automorphisms L_0, L_1 that satisfy

$$L_0(A) = A, \quad L_0(A^*) = A^* + \frac{[A, [A, A^*]_q]}{(q - q^{-1})(q^2 - q^{-2})}$$

$$L_1(A^*) = A^*, \quad L_1(A) = A + \frac{[A^*, [A^*, A]_q]}{(q - q^{-1})(q^2 - q^{-2})}$$

$$L_0^{-1}(A) = A, \quad L_0^{-1}(A^*) = A^* + \frac{[A, [A, A^*]_{q^{-1}}]}{(q - q^{-1})(q^2 - q^{-2})}$$

$$L_1^{-1}(A^*) = A^*, \quad L_1^{-1}(A) = A + \frac{[A^*, [A^*, A]_{q^{-1}}]}{(q - q^{-1})(q^2 - q^{-2})}$$

pf This result was first established by computer by P. Baseilhac and S. Koele.

A computer-free proof was given in

P. Terwilliger. The Lusztig automorphism of the q -Onsager algebra.

□

Comparing Lem 64, Thm 65 we see that

the following diagrams commute:

$$\begin{array}{ccc}
 \Theta_f & \xrightarrow{\hookrightarrow} & \Delta_f \\
 \downarrow L_0^{\pm 1} & & \downarrow T_0^{\pm 1} \\
 \Theta_f & \xrightarrow[\hookrightarrow]{} & \Delta_f
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Theta_f & \xrightarrow{\hookrightarrow} & \Delta_f \\
 \downarrow L_1^{\pm 1} & & \downarrow T_1^{\pm 1} \\
 \Theta_f & \xrightarrow[\hookrightarrow]{} & \Delta_f
 \end{array}$$

Until further notice, let V denote a f.d. mod $\mathcal{O}_f$ -module mod which A, A^* are diagonalizable.

So A, A^* act on V as a tridiagonal pair

We will show that

$$L_0(A), L_0(A^*)$$

(★)

and

$$L_1(A), L_1(A^*)$$

(★★)

act on V as tridiagonal pairs that are isomorphic

to A, A^* . We will focus on (★), the case

of (★★) is similar.

Fix a TD system on V associated with A, A^* .

$$\Phi = (A; \{E_i\}_{i=0}^d, A^*; \{E_i^*\}_{i=0}^d)$$

By const $\exists$ nmo $a, b \in \mathbb{F}$ st

$$\theta_i = a q^{d-2i} + a^* q^{2i-d} \quad (0 \leq i \leq d)$$

$$\theta_i^* = b q^{d-2i} + b^* q^{2i-d}$$

Consider any $H \in \text{End}(V)$ of form

$$H = \sum_{i=0}^d t_i E_i \quad 0 \neq t_i \in \mathbb{F} \quad t_0 = 1$$

H is invertible and

$$H^{-1} = \sum_{i=0}^d t_i^{-1} E_i$$

Obs

$$H^{-1} A H = A$$

LEM 66 With above notation, TFAE

$$(i) \quad L_0(A^*) = H^{-1} A^* H$$

(ii) For $0 \leq i, j \leq d$ st $|i-j|=1$,

$$\frac{t_j}{t_i} = 1 + \frac{(\theta_i - \theta_j)(q\theta_i - q^{-1}\theta_j)}{(q - q^{-1})(q^2 - q^{-2})}$$

pf obs

$$H^{-1} A^* H = \sum_{i=0}^d \sum_{j=0}^d t_i^{-1} t_j E_i A^* E_j$$

and

$$L_0(A^*) = \sum_{i=0}^d \sum_{j=0}^d E_i L_0(A^*) E_j$$

$$= \sum_{i=0}^d \sum_{j=0}^d E_i \left(A^* + \frac{[A, [A, A^*]_q]}{(q - q^{-1})(q^2 - q^{-2})} \right) E_j$$

$$= \sum_{i=0}^d \sum_{j=0}^d E_i A^* E_j \left(1 + \frac{(\theta_i - \theta_j)(q\theta_i - q^{-1}\theta_j)}{(q - q^{-1})(q^2 - q^{-2})} \right)$$

The result follows

□

LEM 67 For $0 \leq i, j \leq d$ s.t. $|i-j|=1$
 the following are multiplicative inverses:

$$1 + \frac{(\theta_i - \theta_j)(q\theta_i - q^{-1}\theta_j)}{(q - q^{-1})(q^2 - q^{-2})},$$

$$1 + \frac{(\theta_j - \theta_i)(q\theta_j - q^{-1}\theta_i)}{(q - q^{-1})(q^2 - q^{-2})}.$$

pf Use

$$\theta_i^2 - (q^2 + q^{-2})\theta_i\theta_j + \theta_j^2 = -(q^2 - q^{-2})^2$$

and algebraic manipulation. □

Using Lem 67 we get the following "improved"
 version of Lem 66.

LEM 68 With above notation TFAE

$$(i) L_0(A^*) = H^* A^* H$$

$$(ii) f_{1 \leq j \leq d}$$

$$\frac{t_j}{t_n} = 1 + \frac{(a_{j+1} - a_j)(q^{2j} \theta_{j+1} - q^{-2j} \theta_j)}{(q - q^{-1})(q^{2n} - q^{-2n})}$$

pf By LEM 66, 67.

□

LEM 69 For $1 \leq j \leq d$,

$$1 + \frac{(a_{j+1} - a_j)(q^{2j} \theta_{j+1} - q^{-2j} \theta_j)}{(q - q^{-1})(q^{2n} - q^{-2n})} = a^2 q^{2(d-2j+1)}$$

pf Use

$$\theta_i = a q^{d-2i} + a^{-1} q^{2i-d} \quad (0 \leq i \leq d)$$

Def 70 For $0 \leq i \leq d$ define

$$t_i = a^{2i} q^{2i(d-i)}$$

We note that $t_0 = 1$ and

$$t_i / t_{i-1} = a^2 q^{2(d-2i+1)} \quad (1 \leq i \leq d)$$

By LEM 68, L9

$$L_0(A^x) = H^T A^x H.$$

Prop 71 For $X \in \mathcal{O}_q$ the following holds
in V :

$$L_0(X) = H^T X H,$$

$$L_0^{-1}(X) = H X H^T$$

pf wlog $X = A$ or $X = A^x$. We saw

$$L_0(A) = A = H^T A H \text{ and } L_0(A^x) = H^T A^x H.$$

□

DEF 72 : let A denote any (assoc) TF-algebra
and let σ denote an aut of A .

let W denote an A -module.

There exists an A -module structure on W ,

called W twisted via σ that behaves as follows.

$\forall x \in A$ and $w \in W$ the vector $x \cdot w$ computed
in W twisted via σ coincides with the vector

$$\sigma^{-1}(x) \cdot w$$

computed in the original A -module W .

We sometimes abbreviate σW for

W twisted via sigma.

Prop 73 the following $\mathcal{O}_g$ -modules are
isomorphic:

(i) the $\mathcal{O}_g$ -module V

(ii) the $\mathcal{O}_g$ -module V twisted via L_0

(iii) the $\mathcal{O}_g$ -module V twisted via L_0^{-1}

Moreover the map H is an iso of $\mathcal{O}_g$ -modules
from (i) to (ii) and from (iii) to (i)

p f By Prop 71 and Def 72

□

Prop 74 The following are isomorphic
triangular systems on V :

$$(i) \quad \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

$$(ii) \quad \Phi^{L_0} = (A, \{E_i\}_{i=0}^d; H^* A^* H, \{H^* E_i^* H\}_{i=0}^d)$$

$$(iii) \quad \Phi^{L_1} = (A, \{E_i\}_{i=0}^d; H A^* H^*; \{H E_i^* H^*\}_{i=0}^d)$$

Moreover, the map H is an iso of TD systems
from (ii) to (i) and from (i) to (iii).

pf By Prop 73 and the construction. $\square$

The map H is discussed further in

P. Terwilliger. Twisting finite-dimensional
modules for the q -Onsager algebra $\mathcal{O}_q$ via the Lusztig
automorphism. The Ramanujan Journal 61 (2023) 175-202