

Math 846

Lecture 4

We continue to discuss a distance-regular graph  $\Gamma = (X, E)$  with diameter  $D \geq 1$ .

Recall the polynomials  $\{v_i\}_{i=0}^{DH}$  from Def 16.

LEM 17 We have

$$(i) \quad v_i(A) = A_i \quad (0 \leq i \leq D)$$

(ii) the minimal polynomial of  $A$  is  $c_1 c_2 \dots c_D v_{DH}$

$$(iii) \quad v_{DH} = \frac{(\lambda - \theta_0)(\lambda - \theta_1) \dots (\lambda - \theta_D)}{c_1 c_2 \dots c_D}$$

pf Consider two sequences:

$$v_0(A), v_1(A), \dots, v_D(A), v_{DH}(A) \quad (*)$$

$$A_0, A_1, \dots, A_D, 0 \quad (**)$$

The sequences  $(*)$ ,  $(**)$  satisfy the same recurrence (Lem 15, Def 16) and

same init conditions  $v_0(A) = A_0 = I$ ,  $v_1(A) = A_1 = A$

So  $(*)$ ,  $(**)$  coincide. Result follows  $\square$

Note For  $\Gamma = H(D, 2)$  the  $\{v_i\}_{i=0}^D$  are called the Krawtchouk polynomials

Adjusting the polynomials  $\{v_i\}_{i=0}^{D-1}$

Def 18 We recursively define some polynomials  $\{u_i\}_{i=0}^{D-1}$  in  $\mathbb{F}[\lambda]$ :

$$u_0 = 1, \quad u_i = \frac{\lambda}{k}$$

$$\lambda u_i = c_i u_{i+1} + a_i u_i + b_i u_{i-1} \quad (1 \leq i \leq D-1)$$

$$\lambda u_D = c_D u_{D+1} + a_D u_D + b_D u_{D-1}$$

obs for  $0 \leq i \leq D$

$u_i$  has degree  $i$  and leading coef  $\frac{1}{b_0 b_1 \dots b_{i-1}}$

$u_{D+1}$  has degree  $D+1$  and leading coef  $\frac{1}{b_0 b_1 \dots b_D}$

LEM 19 We have

$$u_i = \frac{v_i}{k_i} \quad (0 \leq i \leq n)$$

$$u_{DM} = \frac{v_{DM}}{k_0} = \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})}{b_0 b_1 \cdots b_{n-1}}$$

pf Use Defs 16, 18 and Lem 17.  $\square$

Recall the max'l eigenvalue  $\theta_0 = k$ .

LEM 20 For  $0 \leq i \leq n$ ,

$$(i) \quad u_i(\theta_0) = 1$$

$$(ii) \quad v_i(\theta_0) = k_i$$

pf (i) By induction on  $i$ . We have  $u_0 = 1$

so  $u_0(\theta_0) = 1$ . We have  $u_i = \frac{\lambda}{k}$  so  $u_i(\theta_0) = \frac{\theta_0}{k} = \frac{k}{k} = 1$

For  $1 \leq i \leq n-1$

$$u_{i+1}(\theta_0) = \frac{\overset{1}{\theta_0} u_i(\theta_0) - \overset{1}{c_i} u_{i-1}(\theta_0) - \overset{1}{a_i} u_i'(\theta_0)}{b_i}$$

$$= \frac{\theta_0 - c_i - a_i}{b_i} = \frac{k - c_i - a_i}{b_i} = \frac{b_i}{b_i} = 1$$

(ii) By LEM 19 and (i).  $\square$

We saw  $\{A_i\}_{i=0}^D$  form a basis for  $M$   
 Next, we describe another basis for  $M$ .

Def 21 For  $0 \leq i \leq D$  define

$$E_i \in \text{Mat}_X(F)$$

st

$E_i$  acts as  $I$  on the  $\theta_i$ -eigenspace of  $A$

$E_i$  acts as  $0$  on the  $\theta_j$ -eigenspace of  $A$   
 $(0 \leq j \leq D, j \neq i)$

We call  $E_i$  the primitive idempotent of  $V$

for  $\theta_i$ .

By linear algebra:

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq n)$$

$$I = \sum_{i=0}^n E_i$$

$$A = \sum_{i=0}^n \theta_i E_i$$

$$A E_i = \theta_i E_i = E_i A \quad (0 \leq i \leq n)$$

$$E_i = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq n)$$

$$\bar{E}_i = E_i \quad (0 \leq i \leq n)$$

$$E_i^t = E_i \quad (0 \leq i \leq n)$$

$$E_i V = \theta_i \text{-eigenspace of } A \quad (0 \leq i \leq n)$$

$$V = \sum_{i=0}^n E_i V \quad (\text{orthog direct sum})$$

$$\{E_i\}_{i=0}^n \text{ is basis for } M,$$

$$\text{Also } E_0 V = \text{span}\{\mathbf{1}\}, \quad E_0 = |X|^{-1} J,$$

Next goal. Find the transition matrices  
between the bases  $\{A_i\}_{i=0}^D$  and  $\{E_i\}_{i=0}^D$

Prop 22  $\quad \quad \quad \text{Fn } 0 \leq j \leq D,$

$$(i) \quad A_j = \sum_{i=0}^D v_j(\theta_i) E_i$$

$$(ii) \quad E_j = |X|^{-1} m_j \sum_{i=0}^D u_i(\theta_j) A_i$$

pf (i)

$$A_j = v_j(A)$$

$$= v_j(A) I$$

$$= v_j(A) \sum_{i=0}^D E_i$$

$$= \sum_{i=0}^D v_j(\theta_i) E_i$$

$$A E_i = \theta_i E_i$$

(ii) Write  $F_j = \sum_{i=0}^p u_i(\theta_j) A_i$

Show

$$E_j = |X|^{-1} m_j F_j$$

Obs

$$F_j \in M$$

First show

$$F_j \in \text{Span}(E_j)$$

Suffices to show

$$A F_j = \theta_j F_j$$

$$A F_j = \sum_{i=0}^p u_i(\theta_j) A A_i$$

$$= u_0(\theta_j) A A_0 + \sum_{i=1}^{p-1} u_i(\theta_j) A A_i + u_p(\theta_j) A A_p$$

$$= u_0(\theta_j) (a_0 A_0 + c_1 A_1) + \sum_{i=1}^{p-1} u_i(\theta_j) (b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}) + u_p(\theta_j) (b_{p-1} A_{p-1} + a_p A_p)$$

$$= A_0 \left( \underbrace{a_0 u_0(\theta_j) + b_{0-1} u_1(\theta_j)}_{\theta_j u_0(\theta_j)} \right) + \sum_{i=1}^{p-1} A_i \left( \underbrace{c_i u_{i-1}(\theta_j) + a_i u_i(\theta_j) + b_{i+1} u_{i+1}(\theta_j)}_{\theta_j u_i(\theta_j)} \right) + A_p \left( \underbrace{c_p u_{p-1}(\theta_j) + a_p u_p(\theta_j)}_{\theta_j u_p(\theta_j)} \right)$$

$$= \theta_j \sum_{i=0}^p u_i(\theta_j) A_i = \theta_j F_j$$

So far

$$F_j \in \text{Span}(E_j)$$

$\exists \alpha \in \mathbb{F}$  st

$$E_j = \alpha F_j$$

To get  $\alpha$ , take trace

$$E_j \text{ is similar to } \left( \begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & \ddots & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right)$$

$$\#1's = m_j$$

$$\text{trace}(E_j) = m_j$$

$F_j$  has all diagonal entries 1

$$\text{trace}(F_j) = |X|$$

So

$$\alpha = |X|^{-1} m_j$$

□

We endow  $\text{Mat}_X(\mathbb{F})$  with a binary operation  $\circ$  called entrywise product

$$\text{F.n. } B, C \in \text{Mat}_X(\mathbb{F})$$

$$(B \circ C)_{x,y} = B_{x,y} C_{x,y} \quad x, y \in X$$

$$\text{F.n. } 0 \leq i, j \leq D$$

$$A_i \circ A_j = \delta_{ij} A_i$$

So  $M$  is closed under  $\circ$

Recall  $\{E_i\}_{i=0}^D$  is basis for  $M$ .

$$\exists \quad q_{ij}^h \in \mathbb{F} \quad (0 \leq h, i, j \leq D)$$

$$\text{sit } E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D)$$

The  $q_{ij}^h$  are called the Krein parameters of  $\Gamma$

obs  $q_{ij}^h \in \mathbb{R}$  since the prim idempotents are real

By construction.

$$q_{ij}^h = q_{ji}^h \quad (0 \leq h, i, j \leq n)$$

Shortly we will show

$$q_{ij}^h \geq 0 \quad (0 \leq h, i, j \leq n)$$

Next, we bring in the dual Bose-Mann algebra.

Until further notice

$$A_X \quad x \in X \quad \text{"the base vertices"}$$

Def 23 For  $0 \leq i \leq n$  define a diagonal matrix  $E_i^* = E_i^*(1) \in \text{Mat}_X(F)$  with  $(i, j)$ -entry

$$(E_i^*)_{ij} = \begin{cases} 1 & \text{if } i(x, j) = i \\ 0 & \text{if } i(x, j) \neq i \end{cases} \quad y \in X$$

We call  $E_i^*$  the  $i$ th dual primitive idempotent  
of  $\Gamma$  wrt  $x$

By construction,

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq 0)$$

$$I = \sum_{i=0}^0 E_i^*$$

$$\overline{E_i^*} = E_i^* \quad (E_i^*)^t = E_i^* \quad (0 \leq i \leq 0)$$

$$E_i^* V = \text{span} \{ \eta^i \mid \eta \in \Gamma_i(x) \}$$

$$V = \sum_{i=0}^0 E_i^* V \quad (\text{orthog. direct sum})$$

$$E_0^* V = \text{span}(\hat{x})$$

DEF 24. The matrices  $\{E_i^*\}_{i=0}^0$  form a basis for a commutative subalgebra of  $\text{Mat}_x(F)$ ,

denoted  $M^* = M^*(x)$ .

We call  $M^*$  the dual Bose-Mesner algebra of  $\Gamma$  wrt  $x$ .