

Math 846

Lecture 39

Recall $\mathbb{F}$ arb $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

universal Askey-Wilson algebra Δ_q .

Next goal: describe how Δ_q is related to $U_q(\mathfrak{sl}_2)$

Def 50 Define $\Lambda \in U_q(\mathfrak{sl}_2)$ by

$$\Lambda = ef(q - q^{-1})^2 + q^{-1}k + qk^{-1}$$

We call Λ the (normalized) Casimir element
of $U_q(\mathfrak{sl}_2)$.

Ex 51. $\Lambda \in \mathbb{Z}(U_q(\mathfrak{sl}_2))$.

Moreover, on the irred $U_q(\mathfrak{sl}_2)$ -module $L(d, \varepsilon)$

$$\Lambda = \varepsilon(q^{d+1} + q^{-d-1})$$

Recall the equitable generators $X^{\pm 1}, Y, Z$
 of $U_q(\mathfrak{sl}_2)$

LEM 52 the Casimir element Λ of $U_q(\mathfrak{sl}_2)$
 is equal to each of

$$qX + q^{-1}Y + qZ - qXqZ$$

$$q^{-1}X + qY + q^{-1}Z - q^{-1}ZqX$$

$$qY + q^{-1}Z + qX - qYqX$$

$$q^{-1}Y + qZ + q^{-1}X - q^{-1}XqY$$

$$qZ + q^{-1}X + qY - qZqY$$

$$q^{-1}Z + qX + q^{-1}Y - q^{-1}YqZ$$

pf Using Lem 16,

$$\Lambda = (q - q^{-1})^2 ef + q^{-1}k + qk^{-1}$$

$$= q^{-1}(1 - XZ)(Y - X^{-1}) + q^{-1}X + qX^{-1}$$

$$= \underbrace{q^{-1}(qZX - q^{-1}XZ - 1 + q^{-1})}_{=0} X^{-1} + q^{-1}X + q^{-1}Y + qZ - q^{-1}XqY$$

$$= q^{-1}X + q^{-1}Y + qZ - q^{-1}XqY$$

obs

$$qX + q^T y + q^T z - q^T yxz$$

$$- \left(qX + q^T y + q^T z - qxyz \right)$$

$$= \underbrace{\left(qxy - q^T yx - q + q^T \right)}_{=0} z$$

$$= 0$$

and

$$qX + q^T y + q^T z - q^T yxz$$

$$- \left(qX + q^T y + q^T z - qyzx \right)$$

$$= y \underbrace{\left(qzx - q^T xz - q + q^T \right)}_{=0}$$

$$= 0$$

Result follows from these comments and symmetry. $\square$

As we investigate $U_1(a_2)$ it is useful to bring in some elements v_x, v_y, v_z .

Observe

$$q(1-yz) = q^*(1-zy)$$

$$q(1-zx) = q^*(1-xz)$$

$$q(1-xy) = q^*(1-yx)$$

Def 53 We define

$$v_x = q(1-yz) = q^*(1-zy)$$

$$v_y = q(1-zx) = q^*(1-xz)$$

$$v_z = q(1-xy) = q^*(1-yx)$$

One checks the following relations:

$$xy = 1 - q^{-1}vz$$

$$yx = 1 - qvz$$

$$\frac{xy - yx}{q - q^{-1}} = vz$$

$$\frac{qyx - q^{-1}xy}{q - q^{-1}} = 1 - (q + q^{-1})vz$$

$$xv_y = q^2 v_y x$$

$$xv_z = q^{-2} v_z x$$

$$v_x x = 1 - qy - q^{-1}z$$

$$xv_x = 1 - q^{-1}y - qz$$

$$\frac{xv_x - v_x x}{q - q^{-1}} = y - z$$

$$1 = \frac{qv_x - q^{-1}v_x q}{q - q^{-1}} + (q + q^{-1})z,$$

$$1 = \frac{qv_x x - q^{-1}xv_x}{q - q^{-1}} + (q + q^{-1})y$$

$$v_x v_y = 1 - q^{-1}1z + q^{-2}z^2,$$

$$v_y v_x = 1 - q1z + q^2 z^2$$

$$\frac{qv_x v_y - q^{-1}v_y v_x}{q - q^{-1}} = 1 - z^2$$

(the permutation $x \rightarrow y \rightarrow z \rightarrow x$ yields additional relations)

Until further notice, let a, b, c
denote mutually commuting indeterminates.

Let $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ denote the $\mathbb{F}$ -algebra
of Laurent polynomials in a, b, c that have all
coefs in $\mathbb{F}$. Consider the $\mathbb{F}$ -algebra

$$U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$$

where

$$U = U_1(\mathfrak{sl}_2) \quad \text{and} \quad \otimes = \otimes_{\mathbb{F}}$$

Thm 54 $\exists$ $\mathbb{F}$ -algebra hom

$$\Delta_f \longrightarrow U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$$

that sends

$$A \longrightarrow x \otimes a + y \otimes a^{-1} + \frac{xy - yx}{1 - y^2} \otimes bc^{-1}$$

$$B \longrightarrow y \otimes b + z \otimes b^{-1} + \frac{yz - zy}{1 - y^2} \otimes ca^{-1}$$

$$C \longrightarrow z \otimes c + x \otimes c^{-1} + \frac{zx - xz}{1 - y^2} \otimes ab^{-1}$$

$$\alpha \longrightarrow 1 \otimes (a + a^{-1}) + 1 \otimes (b + b^{-1})(c + c^{-1})$$

$$\beta \longrightarrow 1 \otimes (b + b^{-1}) + 1 \otimes (c + c^{-1})(a + a^{-1})$$

$$\gamma \longrightarrow 1 \otimes (c + c^{-1}) + 1 \otimes (a + a^{-1})(b + b^{-1})$$

pf let $A^v, B^v, C^v, \alpha^v, \beta^v, \gamma^v$ denote the

expressions on the right in the above equations.

One checks that $A^v, B^v, C^v, \alpha^v, \beta^v, \gamma^v$ satisfy the relations in Def 40. This checking is done using the rels below Def 53.

□

THM 55 The map from THM 54 sends

$$\Omega \rightarrow 1 \otimes (a+a^{-1})^2 - 1 \otimes (b+b^{-1})^2 - 1 \otimes (c+c^{-1})^2 - \Lambda^2 \otimes 1$$

pf Use Lem 45 and Thm 54 and the relations below Def 53. □

Note The map in Thm 54 is injective.

This is proven in:

P. Terwilliger. The universal Askey-Wilson algebra and the equitable presentation of $U_q(\mathfrak{sl}_2)$.

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Next goal: describe how Δ_q is related
to the double affine Hecke algebra of type
($C_1^\vee, C_1$).

Assume $\mathbb{F}$ arb, $0 \neq q \in \mathbb{F}$, $q^4 \neq 1$

DEF 56 Let $\hat{H}_q$ denote the (assoc) $\mathbb{F}$ -algebra
defined by generators

$$t_i, t_i^{-1} \quad i \in \Pi \quad \Pi = \{0, 1, 2, 3\}$$

and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \Pi$$

$$t_i + t_i^{-1} \text{ is central in } \hat{H}_q \quad i \in \Pi$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

We call $\hat{H}_q$ the (universal) double affine
Hecke algebra of type $(C_1^\vee, C_1)$

As a warmup, we consider some automorphisms of $\hat{H}_4$.

By Def 56, the scalar q^7 is equal to each of

$$t_0 t_1 t_2 t_3, \quad t_1 t_2 t_3 t_0, \quad t_2 t_3 t_0 t_1, \quad t_3 t_0 t_1 t_2$$

Moreover, $\exists$ aut of $\hat{H}_4$ that sends

$$t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_0$$

DEF 57 The Artin braid group B_3 is defined by generators ρ, σ and relations

$$\rho^3 = \sigma^2$$

For notational convenience define

$$\gamma = \rho^3 = \sigma^2$$

$\exists$ group hom $B_3 \rightarrow \text{PSL}_2(\mathbb{Z})$ that sends

$$\rho \rightarrow p, \quad \sigma \rightarrow s.$$

Via this group hom we pull back the $PSL_2(\mathbb{Z})$ action on Δ_g to get a B_3 -action on Δ_g as a group of automorphisms. This B_3 action is described as follows.

LEM 58 The group B_3 acts on Δ_g as a group of automorphisms such that $\tau(h) = h \ \forall h \in \Delta_g$ and ρ, σ do the following

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + \frac{AB-BA}{1-q^2}$	β	α	γ

pf B_3 m43 and the construction.

□

LEM 59 The group B_3 acts on $\hat{H}_9$

as a group of automorphisms such that

$$\tau(h) = t_0^{-1} h t_0 \quad \forall h \in \hat{H}_9$$

and ρ, σ do the following:

h	t_0	t_1	t_2	t_3
$\rho(h)$	t_0	$t_0^{-1} t_3 t_0$	t_1	t_2
$\sigma(h)$	t_0	$t_0^{-1} t_3 t_0$	$t_1 t_2 t_1^{-1}$	t_1

p f This is routinely checked.

□

LEM 60 The B_3 action on $\hat{H}_1$ does
the following to the central elements

$$t_i + t_i^{-1} \quad i \in \mathbb{I}$$

We have

$$\tau(t_i + t_i^{-1}) = t_i + t_i^{-1} \quad i \in \mathbb{I}$$

and

h	$t_0 + t_0^{-1}$	$t_1 + t_1^{-1}$	$t_2 + t_2^{-1}$	$t_3 + t_3^{-1}$
$\rho(h)$	$t_0 + t_0^{-1}$	$t_3 + t_3^{-1}$	$t_1 + t_1^{-1}$	$t_2 + t_2^{-1}$
$\sigma(h)$	$t_0 + t_0^{-1}$	$t_3 + t_3^{-1}$	$t_2 + t_2^{-1}$	$t_1 + t_1^{-1}$

pf By Lem 59

□

THM 61 $\exists$ IF-algebra hom

$$\psi: \Delta_q \rightarrow \hat{H}_q$$

that sends

$$A \rightarrow t_1 t_0 + (t_1 t_0)^{-1}$$

$$B \rightarrow t_3 t_0 + (t_3 t_0)^{-1}$$

$$C \rightarrow t_2 t_0 + (t_2 t_0)^{-1}$$

$$\alpha \rightarrow (t_1^{-1} t_0 + t_1 t_0^{-1})(t_1 + t_1^{-1}) + (t_2 + t_2^{-1})(t_3 + t_3^{-1})$$

$$\beta \rightarrow (t_1^{-1} t_0 + t_1 t_0^{-1})(t_3 + t_3^{-1}) + (t_1 + t_1^{-1})(t_2 + t_2^{-1})$$

$$\gamma \rightarrow (t_1^{-1} t_0 + t_1 t_0^{-1})(t_2 + t_2^{-1}) + (t_3 + t_3^{-1})(t_1 + t_1^{-1})$$

Pf let $A'', B'', C'', \alpha'', \beta'', \gamma''$ denote the expressions on the right in the above display. One checks that $A'', B'', C'', \alpha'', \beta'', \gamma''$ satisfy the relations in Def 40.

□

THM 62 The map ψ from Thm 61 sends

$$\Omega \rightarrow (q+q^{-1})^2 - (q^{-1}t_0 + q t_0^{-1})^2 - (t_1 + t_1^{-1})^2 \\ - (t_2 + t_2^{-1})^2 - (t_3 + t_3^{-1})^2 - (q^{-1}t_0 + q t_0^{-1})(t_1 + t_1^{-1})(t_2 + t_2^{-1})(t_3 + t_3^{-1}).$$

pf Use Lem 45 and Thm 61. □

THM 63 $\forall g \in B_3$ the following diagram commutes:

$$\begin{array}{ccc} \Delta_7 & \xrightarrow{\psi} & \hat{H}_7 \\ g \downarrow & & \downarrow g \\ \Delta_7 & \xrightarrow[\psi]{} & \hat{H}_7 \end{array}$$

pf WLOG $g = \rho$ or $g = \sigma$. To verify the commuting diagram in each case, compare Lem 58, 59. □

Note

For the map $\psi: \Delta_g \rightarrow \hat{H}_g$ in Thm 61

the injectivity is proven in

P. Terwilliger: The Universal Askey-Wilson
algebra and DAHA of type $(C_1^\vee, C_1)$.

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