

Math 846

Lecture 38

Our next topic is the
Universal Askey Wilson Algebra

As a motivation, we start with a Leonard
system of q -Racah type

Until further notice, assume

$$\mathbb{F} \text{ arb, } d \geq 2$$

$$0 \neq q, a, b, c \in \mathbb{F} \quad s.t.$$

$$(i) \quad q^{2i} \neq 1 \quad 1 \leq i \leq d$$

$$(ii) \quad \text{Neither of } a^2, b^2 \text{ is among } q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$$

$$(iii) \quad \text{None of } abc, a^{-1}b, ab^{-1}c, abc^{-1} \text{ is among } q^{d-1}, q^{d-3}, \dots, q^{1-d}$$

We will use q, a, b, c to define a parameter
array.

DEF 36 let

$$\theta_i = a q^{2i-d} + a^{-1} q^{d-2i}$$

$$\theta_i^{-1} = b q^{2i-d} + b^{-1} q^{d-2i}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= a^{-1} b^{-1} q^{d+1} (q^i - q^{-i}) (q^{i-d-1} - q^{d-i-1}) \\ &\quad \times (q^{-i} - a b c q^{i-d-1}) (q^{-i} - a b c^{-1} q^{i-d-1}) \end{aligned}$$

$$\begin{aligned} \phi_i &= a b^{-1} q^{d+1} (q^i - q^{-i}) (q^{i-d-1} - q^{d-i-1}) \\ &\quad \times (q^{-i} - a^{-1} b c q^{i-d-1}) (q^{-i} - a^{-1} b c^{-1} q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$.

LEM 37 The sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^{-1}\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

is a parameter array (of q -Racah type.)

pf One routinely checks PA1-PA5 in Thm 82

□

Let V denote a vector space over $\mathbb{F}$ with $\dim d+1$

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$

denote a Leonard system on V with the parameter array from Lem 37.

Recall that in the basis for V_1

$$A: \begin{pmatrix} \theta_0 & & & & \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & & & \\ & \theta_1^* & \phi_2 & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \phi_d \\ & & & & & \theta_d^* \end{pmatrix}$$

and in another basis for V_1

$$A: \begin{pmatrix} \theta_d & & & & \\ & \theta_{d-1} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_0 \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & & & \\ & \theta_1^* & \phi_2 & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \phi_d \\ & & & & & \theta_d^* \end{pmatrix}$$

For notational convenience, from now on we write

$$B = A^*$$

Prop 38 $\exists C \in \text{End}(V)$ st

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(a+a^{-1})(q^{d/2} + q^{-d/2}) + (b+b^{-1})(c+c^{-1})}{q+q^{-1}}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(b+b^{-1})(q^{d/2} + q^{-d/2}) + (c+c^{-1})(a+a^{-1})}{q+q^{-1}}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(c+c^{-1})(q^{d/2} + q^{-d/2}) + (a+a^{-1})(b+b^{-1})}{q+q^{-1}}$$

pf Recall that A, B satisfy the Askey-Wilson relations. The present equations are the Askey-Wilson relations in disguise. To reveal the disguise, eliminate C in the first two equations above using the third equation above. □

the above equations are called the E_3 -symmetric Askey-Wilson relations.

The following algebra is motivated by Prop 38.

DEF 39. Fix $0 \neq q \in \mathbb{F}$ st $q^4 \neq 1$.

Let Δ_q denote the (associative) $\mathbb{F}$ -algebra with generators A, B, C and relations that assert each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}},$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ_q .

We call Δ_q the universal Askey-Wilson algebra.

DEF 40 Define $\alpha, \beta, \gamma \in \Delta_q$ such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{1 + q^{-1}}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{1 + q^{-1}}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{1 + q^{-1}}$$

Note that α, β, γ are central in Δ_q .

Note 41 It turns out that the following is a basis for Δ_q :

$$A^i B^j C^k \alpha^r \beta^s \gamma^t \quad i, j, k, r, s, t \in \mathbb{N}$$

Recall the modular group $PSL_2(\mathbb{Z})$ has a presentation by generators p, s such that

$$p^3 = 1, \quad s^2 = 1.$$

Next goal: describe how $PSL_2(\mathbb{Z})$ acts on Δ_f as a group of automorphisms.

To reach the goal, we display an aut of Δ_f that has order 3, and an aut of Δ_f that has order 2.

By construction, Δ_f has an aut that sends

$$A \rightarrow B \rightarrow C \rightarrow A.$$

this aut has order 3.

To find an aut of Δ_f that has order 2, we bring in another presentation of Δ_f

LEM 41 The IF-algebra Δ_q is generated

by A, B, Y . Moreover

$$C = \frac{Y}{1+q^2} - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

$$\alpha = \frac{B^2A - (q^2 + q^{-2})BAB + AB^2 + (q^2 - q^{-2})^2 A + (q - q^{-1})^2 BY}{(q - q^{-1})(q^2 - q^{-2})}$$

$$\beta = \frac{A^2B - (q^2 + q^{-2})ABA + BA^2 + (q^2 - q^{-2})^2 B + (q - q^{-1})^2 AY}{(q - q^{-1})(q^2 - q^{-2})}$$

pf In Def 40, solve for C, α, β in terms of A, B, Y . $\square$

LEM 42 The IF-algebra Δ_q has a presentation by generators A, B, γ and relations

$$A^3 B - [3], A^2 B A + [3], A B A^2 - B A^3 = (q^2 - q^{-2})^2 (B A - A B),$$

$$B^3 A - [3], B^2 A B + [3], B A B^2 - A B^3 = (q^2 - q^{-2})^2 (A B - B A),$$

$$\gamma A = A \gamma,$$

$$\gamma B = B \gamma,$$

$$A^2 B^2 - B^2 A^2 + (q^2 + q^{-2})(B A B A - A B A B) = (q - q^{-1})^2 (B A - A B) \gamma$$

pf Use LEM 41 to express the original defining relations for Δ_q in terms of A, B, γ . □

By LEM 42, Δ_q has an automorphism that sends

$$A \rightarrow B, \quad B \rightarrow A, \quad \gamma \rightarrow \gamma$$

This aut has order 2.

THM 43 The group $PSL_2(\mathbb{Z})$ acts on Δ_g as a group of automorphisms in the following way.

u	A	B	C	α	β	γ
$p(u)$	B	C	A	β	γ	α
$s(u)$	B	A	$C + \frac{AB-BA}{1-\tau^2}$	β	α	γ

p.f. The assertions about $p(u)$ are clear.

Consider $s(u)$. By Lem 42 $\exists$ aut S of Δ_g that

sends $A \rightarrow B, B \rightarrow A, \gamma \rightarrow \gamma$.

Obs $S^2 = 1$. By LEM 41 the map S sends

$\alpha \rightarrow \beta, \beta \rightarrow \alpha, C \rightarrow C + \frac{AB-BA}{1-\tau^2}$.

Result follows. $\square$

Note 44 It turns out, the $PSL_2(\mathbb{Z})$ action on Δ_g from Thm 43 is faithful.

Next we consider the center $Z(\Delta_q)$.

Recall $\alpha, \beta, \gamma \in Z(\Delta_q)$. We now introduce another central element Ω , called the Casimir element.

LEM 45 The following elements of Δ_q coincide:

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^7B\beta - qC\gamma,$$

$$qBCA + q^2A^2 + q^2B^2 + q^{-2}C^2 - qA\alpha - qB\beta - q^7C\gamma,$$

$$qCAB + q^{-2}A^2 + q^2B^2 + q^2C^2 - q^7A\alpha - qB\beta - qC\gamma,$$

$$q^7CBA + q^{-2}A^2 + q^2B^2 + q^{-2}C^2 - q^7A\alpha - qB\beta - q^7C\gamma,$$

$$q^7ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^7A\alpha - q^7B\beta - qC\gamma,$$

$$q^7BAC + q^2A^2 + q^{-2}B^2 + q^{-2}C^2 - qA\alpha - q^7B\beta - q^7C\gamma.$$

We denote this common element by Ω .

pf Apply the relations in Def 40 and the
ant ρ from Thm 43.

() detail:

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

$$- \left(q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma \right)$$

$$= (q - q^{-1})A \underbrace{\left((q + q^{-1})A + \frac{qBC - q^{-1}CB}{q - q^{-1}} \right)}_{=0} - \alpha$$

$$= 0.$$

Also

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

$$- \left(q^{-1}BAC + q^2A^2 + q^{-2}B^2 + q^{-2}C^2 - qA\alpha - q^{-1}B\beta - q^{-1}C\gamma \right)$$

$$= (q - q^{-1}) \underbrace{\left((q + q^{-1})C + \frac{qAB - q^{-1}BA}{q - q^{-1}} \right)}_{=0} - \gamma$$

$$= 0$$

□

(1) THM 46 $\Omega \in Z(\Delta_q)$.

pf We show $\Omega A = A \Omega$. Obs

$$0 = q \left(B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} - \frac{\beta}{q + q^{-1}} \right)$$

$$+ \left(B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} - \frac{\beta}{q + q^{-1}} \right) q^{-1}C$$

$$- \gamma \left(B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} - \frac{\beta}{q + q^{-1}} \right)$$

$$+ \beta \left(C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} - \frac{\gamma}{q + q^{-1}} \right)$$

$$- q^{-1}B \left(C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} - \frac{\gamma}{q + q^{-1}} \right)$$

$$- \left(C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} - \frac{\gamma}{q + q^{-1}} \right) qB$$

$$= \left(qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma \right) \frac{A}{q^2 - q^{-2}}$$

$$- \frac{A}{q^2 - q^{-2}} \left(qBCA + q^2A^2 + q^2B^2 + q^{-2}C^2 - qA\alpha - qB\beta - q^{-1}C\gamma \right)$$

$$= \frac{\Omega A - A \Omega}{q^2 - q^{-2}}$$

This shows $\Omega A = A \Omega$.

One similarly shows $\Omega B = B \Omega$ and $\Omega C = C \Omega$

□

Thm 47 Ω is fixed by everything
in $PSL_2(\mathbb{Z})$.

pf The auts p and s both permute the m^*
expressions in Lem 45. □

Note 48 It turns out that if q is not a root of unity,
then the center $\mathbb{Z}(\Delta_q)$ is generated by $\alpha, \beta, \gamma, \Omega$. Moreover
 $\alpha, \beta, \gamma, \Omega$ are algebraically independent.

Prop 49 $\exists$ $\mathbb{F}$ -algebra hom $\psi: \mathcal{O}_q \rightarrow \Delta_q$

that sends

$$A \rightarrow A,$$

$$A^* \rightarrow B$$

pf By Lem 42. □

We will return to Prop 49 shortly.