

Math 846

Lecture 37

(1) Until further notice, assume

$$\mathbb{F} \text{ arb} \quad 0 \neq q \in \mathbb{F}, \quad q^4 \neq 1$$

Recall the q -Onsager algebra $\mathcal{O}_q$ has generators A, A^* and relations

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A],$$

$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*]$$

Our next goal is to describe how $\mathcal{O}_q$ is related to

$$\boxtimes_q \text{ and } U_q(\mathfrak{sl}_2).$$

We will prove the following two theorems.

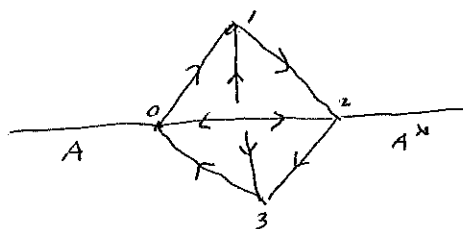
Thm 24 For nonzero $a, b \in \mathbb{F}$ $\exists$ algebra hom

$$\mathcal{O}_q \rightarrow \boxtimes_q$$

that sends

$$A \rightarrow a x_{30} + a^{-1} x_{01},$$

$$A^* \rightarrow b x_{12} + b^{-1} x_{23}$$



(1) Thm 25 For $\alpha \neq 0$ $a, b \in F$ $\exists$ algebra hom

$$\mathcal{O}_q \rightarrow U_q(\hat{\mathfrak{sl}}_2)$$

that sends

$$A \rightarrow a y_0 + a^* z_0, \quad A^* \rightarrow b y_1 + b^* z_1$$

Note In thm 24, 25 the map is injective, but the proof of this fact is lengthy.

Def 26 let $\tilde{\square}_q$ denote the (assoc) F -algebra

with generators

$$x_i, c_i, c_i^{-1} \quad (i \in \mathbb{Z}_n)$$

and relations

$$c_i c_i^{-1} = c_i^{-1} c_i = 1$$

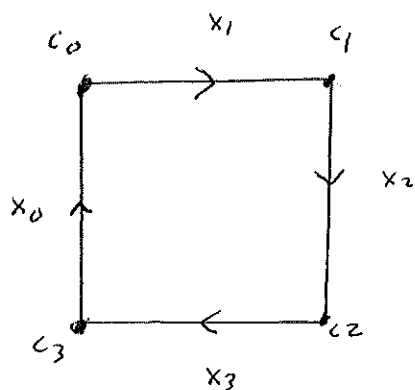
$$c_i^{\pm 1} \text{ is central in } \tilde{\square}_q$$

$$(i \in \mathbb{Z}_n)$$

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = c_i$$

Informal
Diagram:

L37 p3



means

$$\frac{qr - r^2 s}{q - r^2} = c$$

Def 27 For the algebra $\tilde{\square}_q$ define

$$S_i = x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3$$

($i \in \mathbb{Z}_4$)

Ex 28 For the algebra $\tilde{\square}_q$ and $i \in \mathbb{Z}_4$,

$$q^2 S_i x_{i+1} = q^{-2} x_{i+1} S_i$$

$$q^{-2} S_i x_{i-1} = q^2 x_{i-1} S_i$$

() LEM 29 For the algebra $\tilde{\Omega}_q$ define

$$A = x_0 + x_1 \quad A^* = x_2 + x_3.$$

then

$$A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 \\ + (q^2 - q^{-2})^2 c_0 (A A^* - A^* A) = S_0 + S_1,$$

$$A^{*3} A - [3]_q A^{*2} A A^* + [3]_q A^* A A^{*2} - A A^{*3} \\ + (q^2 - q^{-2})^2 c_2 (A^* A - A A^*) = S_2 + S_3.$$

pf Routine (but tedious) using the defining relations
for $\tilde{\Omega}_q$ from Def 26. □

() DEF 30 let $\hat{\square}_q$ denote the unsoal $\mathbb{F}$ -algebra

with generators

$$x_i, c_i, c_i^{-1} \quad (i \in \mathbb{Z}_4)$$

and relations

$$c_i c_i^{-1} = c_i^{-1} c_i = 1$$

$$c_i^{\pm 1} \text{ is central in } \hat{\square}_q$$

$$(i \in \mathbb{Z}_4)$$

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = c_i$$

$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0$$

Def 31 By construction $\exists \mathbb{F}$ -alg hom

$$\tilde{\square}_q \rightarrow \hat{\square}_q$$

that sends

$$x_i \rightarrow x_i$$

$$(i \in \mathbb{Z}_4)$$

$$c_i^{\pm 1} \rightarrow c_i^{\pm 1}$$

call this map canonical

() LEM 32 For the algebra $\hat{\square}_q$ define

$$A = x_0 + x_1$$

$$A^* = x_2 + x_3$$

Then

$$\begin{aligned} A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 \\ = c_0 (q^2 - q^{-2})^2 (A^* A - A A^*), \end{aligned}$$

$$\begin{aligned} A^* A^3 - [3]_q A^* A^2 A + [3]_q A^* A A^2 - A A^* A^3 \\ = c_2 (q^2 - q^{-2})^2 (A A^* - A^* A). \end{aligned}$$

p f Apply the canonical map $\tilde{\square}_q \rightarrow \hat{\square}_q$
to everything in Lem 29. □

LEM 33 For $i \in \mathbb{Z}_4$ let α_i denote
 an invertible central element in $\hat{\Delta}_q$. Then
 $\exists$ automorphism of $\hat{\Delta}_q$ that sends

$$x_i \rightarrow \alpha_i x_i$$

$$c_i \rightarrow \alpha_i \alpha_{i+2} c_i$$

$$c_i^{-1} \rightarrow \alpha_i^{-1} \alpha_{i+2}^{-1} c_i^{-1}$$

$$(i \in \mathbb{Z}_4)$$

pf Routine using def 30.

□

LEM 34. For the algebra $\hat{\square}_q$ define

$$A = \alpha_0 x_0 + \alpha_1 x_1$$

$$A^* = \alpha_2 x_2 + \alpha_3 x_3$$

where $\{\alpha_i\}_i \in \mathbb{Z}_q$ are from Lem 33. Then

$$\begin{aligned} A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 \\ = \alpha_0 \alpha_1 c_0 (q^2 - q^{-2})^2 (A^* A - A A^*) \end{aligned}$$

$$\begin{aligned} A^{*3} A - [3]_q A^{*2} A A^* + [3]_q A^* A A^{*2} - A A^{*3} \\ = \alpha_2 \alpha_3 c_2 (q^2 - q^{-2})^2 (A A^* - A^* A) \end{aligned}$$

pf Apply the ant from Lem 33 to everything in Lem 32.

LEM 35 Let $\{c_i\}_{i \in \mathbb{Z}_4}$ denote invertible central elements in $\hat{\square}_7$ st

$$c_0 c_1 c_2 = 1,$$

$$c_2 c_3 c_2 = 1.$$

Then $\exists$ algebra hom

$$\theta_g \rightarrow \hat{\square}_7$$

that sends

$$A \rightarrow c_0 x_0 + c_1 x_1$$

$$A^b \rightarrow c_2 x_2 + c_3 x_3$$

pf By Lem 34.

□

pf of Th 24.

For nmo $a, b \in F$ we display

an algebra hom

 $\theta_f \rightarrow \boxtimes_f$ that sends

$$A \rightarrow a x_{30} + a^{-1} x_{01}$$

$$A^* \rightarrow b x_{12} + b^{-1} x_{23}$$

In LEM 35 take

$$\alpha_0 = a c_0^{-1}$$

$$\alpha_1 = a^{-1}$$

$$\alpha_2 = b c_2^{-1}$$

$$\alpha_3 = b^{-1}$$

The desired map is the composition of algebra homs

$$\begin{array}{ccc} \theta_f & \longrightarrow & \hat{\square}_f \\ & & \longrightarrow \quad \boxtimes_f \\ \begin{array}{l} A \rightarrow a c_0^{-1} x_0 + a^{-1} x_1 \\ A^* \rightarrow b c_2^{-1} x_2 + b^{-1} x_3 \end{array} & & \begin{array}{l} x_i \rightarrow x_{i+1} \\ c_i \rightarrow 1 \end{array} \end{array}$$

() Pf of Thm 25 For $n \geq 0$ $a, b \in F$ we display
an algebra hom $\mathcal{O}_f \rightarrow U_q(\hat{\mathfrak{sl}}_2)$ that sends

$$A \rightarrow a\gamma_0 + a^{-1}\gamma_1, \quad A^* \rightarrow b\gamma_1 + b^{-1}\gamma_0$$

In Lem 35 take

$$\alpha_0 = a c_0^{-1}$$

$$\alpha_1 = a^{-1}$$

$$\alpha_2 = b c_2^{-1}$$

$$\alpha_3 = b^{-1}$$

The desired map is the composition of algebra homs

$$\mathcal{O}_f \rightarrow \hat{\square}_p \rightarrow U_q(\hat{\mathfrak{sl}}_2)$$

$$A \rightarrow a c_0^{-1} x_0 + a^{-1} x_1$$

$$x_0 \rightarrow \gamma_0$$

$$x_1 \rightarrow \gamma_1$$

$$A^* \rightarrow b c_2^{-1} x_2 + b^{-1} x_3$$

$$x_2 \rightarrow \gamma_2$$

$$x_3 \rightarrow \gamma_3$$

$$c_0 \rightarrow 1$$

$$c_1 \rightarrow x_0^{-1} x_1^{-1}$$

$$c_2 \rightarrow 1$$

$$c_3 \rightarrow x_0^{-1} x_1^{-1}$$

□

For thm 24, 25 the injectivity is shown in:

P. Terwilliger. The q -Onsager algebra and
the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$, Linear Algebra
and its Applications, 521 (2017) 19-56