

Math 846

Lecture 36

DEF 15 Referring to LEM 14, we call
 ϵ the type of the module.

We now give another presentation of $U_q(\mathfrak{sl}_2)$, said
 to be equitable.

LEM 16 The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to
 the algebra with generators

$$x, x^{-1}, y, z$$

and relations

$$x x^{-1} = x^{-1} x = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

(1) An isomorphism with the presentation in DEF 13
is given by

$$x^{\pm 1} \rightarrow k^{\pm 1}$$

$$y \rightarrow k^{-1} + f$$

$$z \rightarrow k^{-1} - k^{-1}e \cdot (1 - q^{-1})^{-2}$$

The inverse isomorphism sends

$$k^{\pm 1} \rightarrow x^{\pm 1}$$

$$e \rightarrow (1 - xz) q^{-1} (1 - q^{-1})^{-2}$$

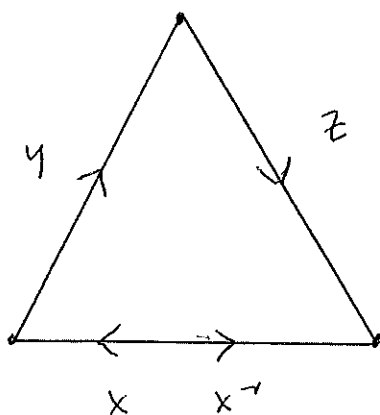
$$f \rightarrow y - x^{-1}$$

pf One checks, each map above is an algebra hom. One checks, these maps are inverses. Therefore, these maps are isomorphisms. $\square$

DEF 17 We call x, x^{-1}, y, z the
equitable generators for $U_q(\mathfrak{sl}_2)$.

A diagram for $U_q(\mathfrak{sl}_2)$:

Represent each equitable generator by a
 directed arc:



Read off relations using the same conventions

as $\boxtimes_f$

LEM 18 For $i \in \mathbb{Z}_4 \quad \exists$ $\mathbb{F}$ -algebra

hom $U_q(\mathfrak{sl}_2) \rightarrow \bigotimes_q$ that sends

$$X \rightarrow X_{i,i+2}$$

$$X^{-1} \rightarrow X_{i+2,i}$$

$$Y \rightarrow X_{i+2,i+3}$$

$$Z \rightarrow X_{i+3,i}$$

pf Routine. □

Next we define $U_q(\hat{\mathfrak{sl}}_2)$. Roughly speaking,

$U_q(\hat{\mathfrak{sl}}_2)$ is generated by 2 copies of $U_q(\mathfrak{sl}_2)$

that are glued together in a certain way.

DEF 19 let $U_q(\hat{\mathfrak{sl}}_2)$ denote the (assoc)

$\mathbb{F}$ -algebra with generators

$$e_i^+, e_i^-, k_i, k_i^{-1} \quad (i=0,1)$$

and relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1$$

$$k_0 k_1 = k_1 k_0$$

$$k_i e_i^{\pm} k_i^{-1} = q^{\pm 2} e_i^{\pm}$$

$$k_i e_j^{\pm} k_i^{-1} = q^{\mp 2} e_j^{\pm} \quad i \neq j$$

$$[e_i^+, e_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$$[r_1, r_2] = r_2 - r_1$$

$$[e_0^{\pm}, e_1^{\mp}] = 0$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0 \quad i \neq j$$

(q-Serre rel)

"Chevalley presentation"

Next we give the equitable presentation of $U_q(\mathcal{A}_2)$

LEM 20 The algebra $U_q(\mathcal{A}_2)$ is isomorphic to the algebra with generators

$$x_i, x_i^{-1}, y_i, z_i \quad (i=0,1)$$

and relations:

$$x_i x_i^{-1} = x_i^{-1} x_i = 1$$

$x_0 x_1$ is central

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1$$

$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1,$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1,$$

$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = x_0^{-1} x_1^{-1} \quad i \neq j$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \quad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \quad i \neq j$$

An iso with the presentation in DEF 19 sends

$$x_i^{\pm 1} \rightarrow k_i^{\pm 1}$$

$$y_i \rightarrow k_i^{-1} + e_i^{-}$$

$$z_i \rightarrow k_i^{-1} - k_i^{-1} e_i^{+} q (q - q^{-1})^{-2}$$

The inverse iso sends

$$k_i^{\pm 1} \rightarrow x_i^{\pm 1}$$

$$e_i^{+} \rightarrow (1 - x_i z_i) q^{-1} (q - q^{-1})^{-2}$$

$$e_i^{-} \rightarrow y_i - x_i^{-1}$$

pf

One checks each map is an algebra hom,

One checks the maps are inverses.

Therefore each map is an algebra iso.

□

let $U_q(L(\mathfrak{sl}_2))$ denote the quotient
 of $U_q(\hat{\mathfrak{sl}}_2)$ by the 2-sided ideal
 generated by $x_0 x_1 - 1$ (in the equitable pres)

the algebra $U_q(L(\mathfrak{sl}_2))$ is called the
 $U_q(\mathfrak{sl}_2)$ loop algebra.

The following result shows how $U_q(L(\mathfrak{sl}_2))$
 is related to $\boxtimes_q$.

LEM 21 The algebra $U_q(L(\mathfrak{sl}_2))$

is iso to the algebra with gens

$$x_i, y_i, z_i \quad (i=0,1)$$

and relations

$$x_0 x_1 = x_1 x_0 = 1,$$

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1,$$

$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1$$

$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = 1 \quad i \neq j$$

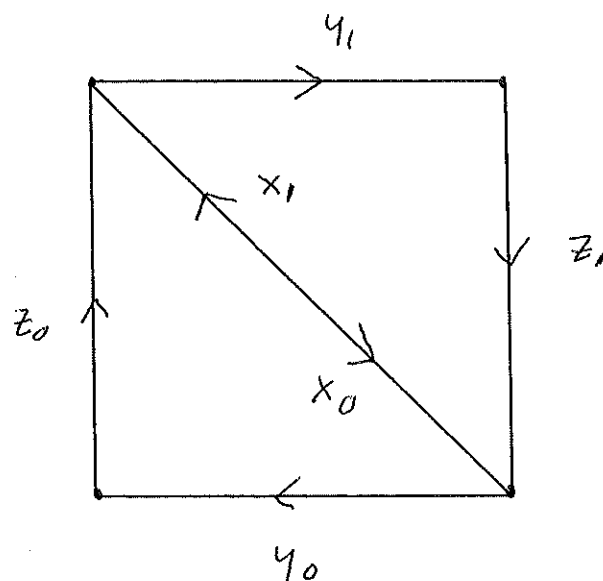
$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \quad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \quad i \neq j$$

pf Routine.

□

Diagram for $U_1(L(st_2))$



Same conventions as for $\boxtimes_z$

(1) Thm 22 For $i \in \mathbb{Z}_4$ $\exists$ alg hom

$$U_q(L(\mathfrak{sl}_2)) \rightarrow \bigotimes_q \text{ that sends}$$

$$X_0 \rightarrow X_{i,i+2} \quad Y_0 \rightarrow X_{i+2,i+3} \quad Z_0 \rightarrow X_{i+3,i}$$

$$X_1 \rightarrow X_{i+2,i} \quad Y_1 \rightarrow X_{i,i+1} \quad Z_1 \rightarrow X_{i+1,i+2}$$

pf clear.

□

(2) Composing the canonical hom $U_q(\hat{\mathfrak{sl}}_2) \rightarrow U_q(L(\mathfrak{sl}_2))$

with the hom in Thm 22, we get an algebra

hom $U_q(\hat{\mathfrak{sl}}_2) \rightarrow \bigotimes_q$. Let V denote a

$\bigotimes_q$ -module. Pulling back the $\bigotimes_q$ -module structure

to $U_q(\hat{\mathfrak{sl}}_2)$ via the above hom, V becomes

a $U_q(\hat{\mathfrak{sl}}_2)$ -module. In particular, the

$\bigotimes_q$ -module from THM 12 becomes a

$U_q(\hat{\mathfrak{sl}}_2)$ -module.

1) COR 23 Referring to our TD system

$\mathbb{F}$ on V , the vector space V becomes a

$U_q(\hat{\mathfrak{sl}}_2)$ -module on which the equitable

generators act as follows

gen	x_0	x_0^{-1}	y_0	z_0	x_1	x_1^{-1}	y_1	z_1
action	k^{-1}	k	A	B	k	k^{-1}	A^*	B^*

gen	x_0	x_0^{-1}	y_0	z_0	x_1	x_1^{-1}	y_1	z_1
action	$(k^*)^{-1}$	k^*	B	A^*	k^*	$(k^*)^{-1}$	B^*	A

Both $U_q(\hat{\mathfrak{sl}}_2)$ -modules are irreducible.

□