

Math 846

Lecture 35

pf, LEM8: We consider each row in the table.

→ Use Lem 3, 4, 6

→ Use Lem 3, 4, 6

↘ View $W_i = (w_0 + \dots + w_i) \cap (w_i + \dots + w_d)$

$$\begin{aligned} (B - q^{2i-d} I) W_i &\subseteq (B - q^{2i-d} I) (w_0 + \dots + w_i) \\ &= (B - q^{2i-d} I) (E_0^* V + \dots + E_i^* V) \\ &\subseteq E_0^* V + \dots + E_i^* V \\ &= w_0 + \dots + w_i \end{aligned}$$

and

$$\begin{aligned} (B - q^{2i-d} I) W_i &\subseteq (B - q^{2i-d} I) (w_i + \dots + w_d) \\ &= (B - q^{2i-d} I) (E_i V + \dots + E_d V) \\ &\subseteq E_i V + \dots + E_d V \\ &= w_i + \dots + w_d \end{aligned}$$

So

$$\begin{aligned} (B - q^{2i-d} I) W_i &\subseteq (w_0 + \dots + w_{i-1}) \cap (w_{i+1} + \dots + w_d) \\ &= W_{i-1} \end{aligned}$$

$\nwarrow$, removed.

$$\begin{aligned}
 (B^* - q^{d-2i} I) W_i &\subseteq (B^* - q^{d-2i} I) (W_0 + \dots + W_i) \\
 &= (B^* - q^{d-2i} I) (E_0^* V + \dots + E_i^* V) \\
 &\subseteq E_0^* V + \dots + E_i^* V \\
 &= W_0 + \dots + W_i
 \end{aligned}$$

and

$$\begin{aligned}
 (B^* - q^{d-2i} I) W_i &\subseteq (B^* - q^{d-2i} I) (W_i + \dots + W_d) \\
 &= (B^* - q^{d-2i} I) (E_i V + \dots + E_d V) \\
 &\subseteq E_i V + \dots + E_d V \\
 &= W_i + \dots + W_d
 \end{aligned}$$

So

$$\begin{aligned}
 (B^* - q^{d-2i} I) W_i &\subseteq (W_0 + \dots + W_i) \cap (W_i + \dots + W_d) \\
 &= W_i
 \end{aligned}$$

$$\{ \quad ; \quad (B - q^{2i-d} I) w_i = 0$$

by the def of B .

Find $B^* w_i$

$$\text{view } w_i = (w_0 + \dots + w_i) \cap (w_i + \dots + w_d)$$

$$\begin{aligned} B^* w_i &\leq B^*(w_0 + \dots + w_i) \\ &= B^*(E_0^* V + \dots + E_i^* V) \\ &\leq E_0^* V + \dots + E_i^* V \\ &= w_0 + \dots + w_i \end{aligned}$$

and

$$\begin{aligned} B^* w_i &\leq B^*(w_i + \dots + w_d) \\ &= B^*(E_i^* V + \dots + E_d^* V) \\ &\leq E_i^* V + \dots + E_d^* V \\ &= w_i + \dots + w_d \end{aligned}$$

So

$$\begin{aligned} B^* w_i &\leq (w_0 + \dots + w_i) \cap (w_i + \dots + w_d) \\ &= w_i + w_i + w_i \end{aligned}$$

() $\begin{smallmatrix} \diagup \\ \diagdown \end{smallmatrix}$ similar to $\begin{smallmatrix} \diagdown \\ \diagup \end{smallmatrix}$

$\begin{smallmatrix} \diagup \\ \diagdown \end{smallmatrix}$ similar to $\begin{smallmatrix} \diagdown \\ \diagup \end{smallmatrix}$



() LEM 9 We have

$$(i) \quad \frac{q B k^{\top} - q^{\top} k^{\top} B}{q - q^{\top}} = I$$

$$(ii) \quad \frac{q B^{\vee} K - q^{\top} K B^{\vee}}{q - q^{\top}} = I$$

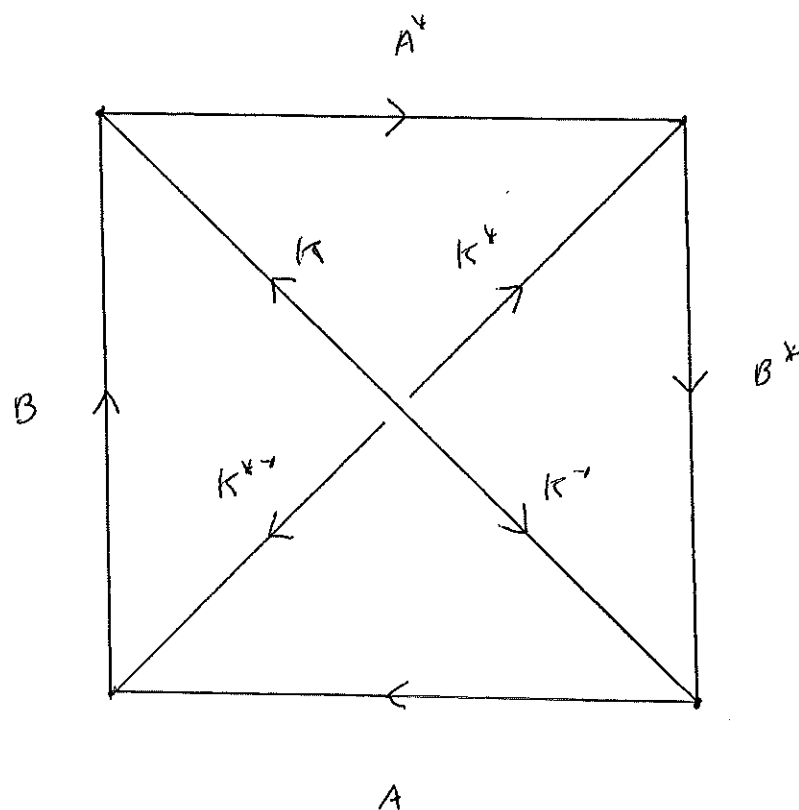
$$(iii) \quad \frac{q K^{\vee \top} B - q^{\top} B K^{\vee \top}}{q - q^{\top}} = I$$

$$(iv) \quad \frac{q K^{\vee} B^{\vee} - q^{\top} B^{\vee} K^{\vee}}{q - q^{\top}} = I$$

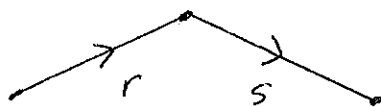
pf Use Lem 3 / Lem 4 and Lem 8

□

the q -Weyl relations in summary



key



means

$$\frac{qrs - q^*sr}{q - q^*} = I$$

A few more relations

Lem 10 We have

$$(i) \quad B^3 B^* - [3], B^2 B^* B + [3], B B^* B^2 - B^* B^3 = 0,$$

$$(ii) \quad B^{*3} B - [3], B^{*2} B B^* + [3], B^* B B^{*2} - B B^{*3} = 0$$

pf Use Lem 5, Lem 8. □

Next, we interpret our results so far in terms
of an algebra

$$\boxtimes_t$$

"q-tetrahedron algebra"

let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ = cyclic group of order 4

DEF 11 For $0 \neq q \in \mathbb{F}$ with $q^2 \neq 1$, let $\boxtimes_q$

denote the (assoc) algebra over $\mathbb{F}$ with generators

$$\{ x_{ij} \mid i, j \in \mathbb{Z}_4, \ i-j=1 \text{ or } i-j=2 \}$$

and relations:

(i) For $i, j \in \mathbb{Z}_4$ s.t. $i-j=2$

$$x_{ij} x_{ji} = 1$$

(ii) For $h, i, j \in \mathbb{Z}_4$ s.t. the pair $(i-h, j-i)$ is one of

$$(1, 1), \quad (1, 2), \quad (2, 1)$$

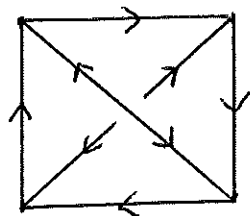
$$\frac{q x_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ s.t. $i-h = j-i = k-j = 1$,

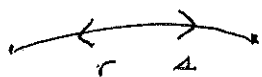
$$x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0$$

(q -Serre relation)

We interpret $\boxtimes_q$ with a diagram:

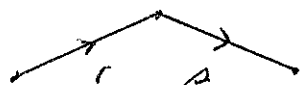


each directed arc represents a generator of $\boxtimes_q$



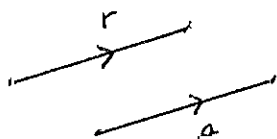
means

$$ra = ar = 1$$



means

$$\frac{qra - r^q ar}{q - q^q} = 1$$



means

$$r^3 a - [3]_q r^2 a r + [3]_q r a r^2 - a r^3 = 0,$$

$$a^3 r - [3]_q a^2 r a + [3]_q a r a^2 - r a^3 = 0$$

The following result is now immediate

Thm 12 Referring to our TD system $\mathbb{F}$ on V
from lec 34,

V becomes a $\mathbb{F}_q$ -module on which the $\mathbb{F}_q$ -generators
act as follows.

gen	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}	x_{20}	x_{31}
action	A^*	B^*	A	B	K^*	K^{*2}	K	K^*

this $\mathbb{F}_q$ -module is irreducible.

□

Note Ref to the TD system $\mathbb{F}$ on V
from Lec 34, the pair B, B^* acts
on V as a TD pair.

Our next goal is to explain how the algebra $\hat{\mathfrak{X}}_q$ is related to $U_q(\hat{\mathfrak{sl}}_2)$.

Before we define $U_q(\hat{\mathfrak{sl}}_2)$, we start with a more basic algebra $U_q(\mathfrak{sl}_2)$.

Until further notice, assume

$$\mathbb{F} \text{ arb} \quad 0 \neq q \in \mathbb{F} \quad q^2 \neq 1$$

DEF 13 Let $U_q(\mathfrak{sl}_2)$ denote the (assoc)

$\mathbb{F}$ -algebra that has generators

$$e, f, k, k^{-1}$$

and relations

$$k k^{-1} = k^{-1} k = 1,$$

$$k e = q^2 e k,$$

$$k f = q^{-2} f k,$$

$$e f - f e = \frac{k - k^{-1}}{q - q^{-1}}$$

Note the above definition is often called
the Chevalley presentation of $U_q(\mathfrak{sl}_2)$.

To get a feel for $U_q(\mathfrak{sl}_2)$, it is helpful to
work out the finite-dim'l uned modules.

EX 14 Assume $\mathbb{F}$ is algebraically closed, and
 q is not a root of 1. Then up to isomorphism
the f.d. uned $U_q(\mathfrak{sl}_2)$ -modules are

$$L(d, \epsilon) \quad d \in \mathbb{N} \quad \epsilon \in \{1, -1\}.$$

The module $L(d, \epsilon)$ has a basis $\{v_i\}_{i=0}^d$ st

$$kv_i = \epsilon q^{d-2i} v_i \quad (0 \leq i \leq d)$$

$$fv_i = [i]_q v_{i-1} \quad (0 \leq i \leq d-1), \quad fv_d = 0$$

$$ev_i = \epsilon [d-i]_q v_{i+1} \quad (1 \leq i \leq d), \quad ev_0 = 0$$

(if $\text{char}(\mathbb{F}) = 2$ we view $\{1, -1\}$ as having a
single element)