

Math 846

Lecture 34

Until further notice:

$$\mathbb{F} \text{ arb} \quad d \in \mathbb{N}$$

$$0 \neq q \in \mathbb{F} \quad q^{2i} \neq 1 \quad (1 \leq i \leq d)$$

$V =$ f.d. vector space over $\mathbb{F}$.

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a TD system on V with eigenvalue sequence

$$\theta_i = q^{2i-d} \quad (0 \leq i \leq d)$$

and dual eigenvalue sequence

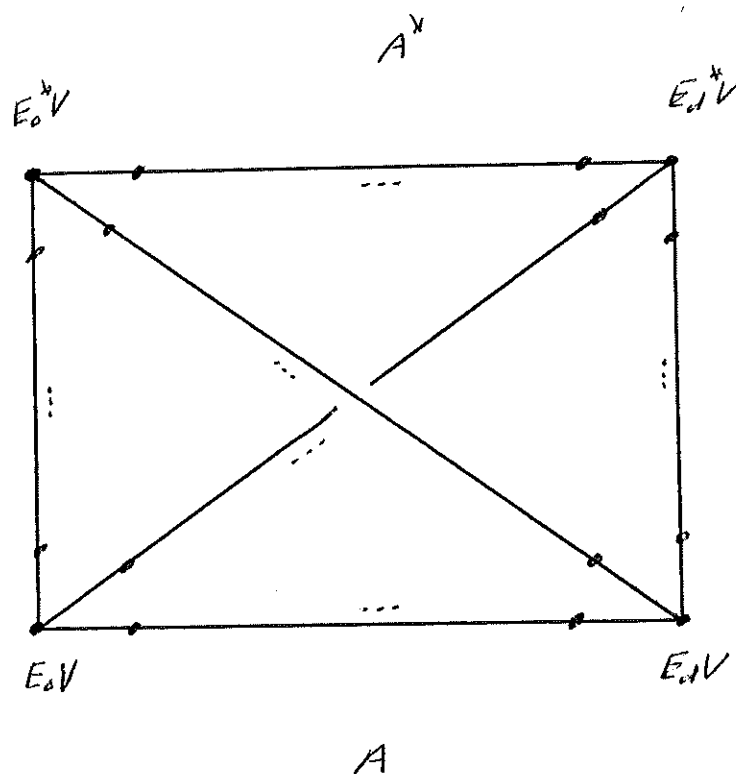
$$\theta_i^* = q^{d-2i} \quad (0 \leq i \leq d)$$

In this case, the TD relations become the q -Serre relations

$$A^3 A^* - [3]_q A^2 A^* A + [3]_1 A A^* A^2 - A^* A^3 = 0,$$

$$A^{*3} A - [3]_1 A^{*2} A A^* + [3]_q A^* A A^{*2} - A A^{*3} = 0$$

Recall the tetrahedron diagram for Φ :



In the table below, we recall the meaning.

We have

decomp	i'th subspace of decomp
	$E_i V$
	$E_i^* V$
	$(E_0^* V + \dots + E_i^* V) \cap (E_i V + \dots + E_d V)$
	$(E_0^* V + \dots + E_i^* V) \cap (E_0 V + \dots + E_{d-i} V)$
	$(E_{d-i}^* V + \dots + E_d^* V) \cap (E_0 V + \dots + E_{d-i} V)$
	$(E_{d-i}^* V + \dots + E_d^* V) \cap (E_i V + \dots + E_d V)$

We now summarize the action of A and A^* on the six decompositions.

LEM 1 Let $\{W_i\}_{i=0}^d$ denote a decomp of V from the above table. Then for each the action of A and A^* on W_i is:

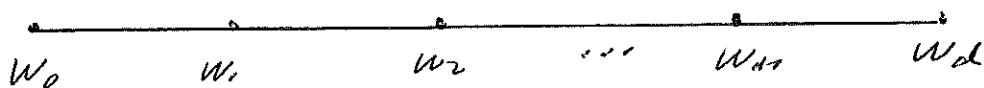
decomp	A action	A^* action
	$(A - \theta_i I) W_i = 0$	$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$
	$A W_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(A^* - \theta_i^* I) W_i = 0$
	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i-1}$
	$(A - \theta_{i+1} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i-1}$
	$(A - \theta_{i+1} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{i+1}^* I) W_i \subseteq W_{i+1}$
	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{i+1}^* I) W_i \subseteq W_{i+1}$

Pf By const of $\boxtimes$ diagram.

□

Notation

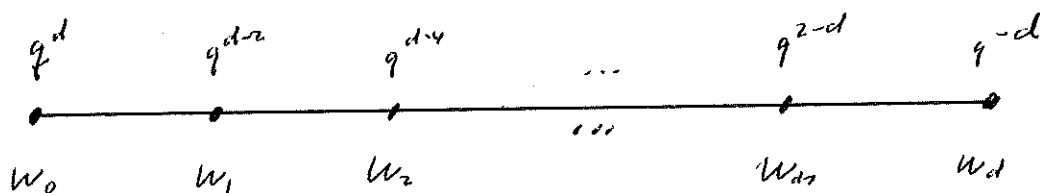
Given a decomp $\{W_i\}_{i=0}^d$ of V :



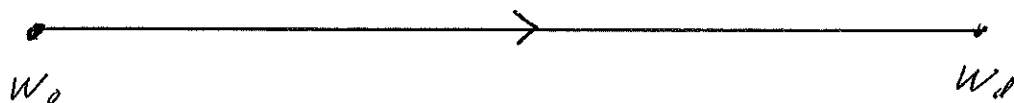
Define a map $T \in \text{End}(V)$ st $\forall 0 \leq i \leq d$,

W_i is an eigenspace of T with eigenvalue q^{d-2i} .

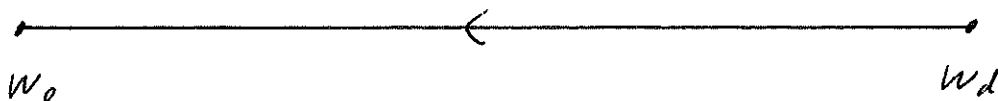
So



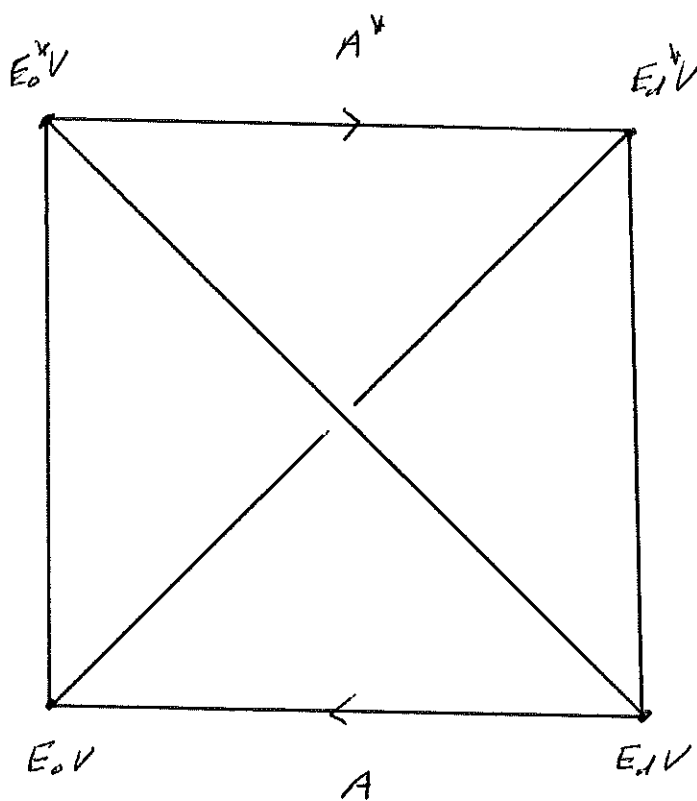
We often represent T by a directed arc



The inverse of T is represented by



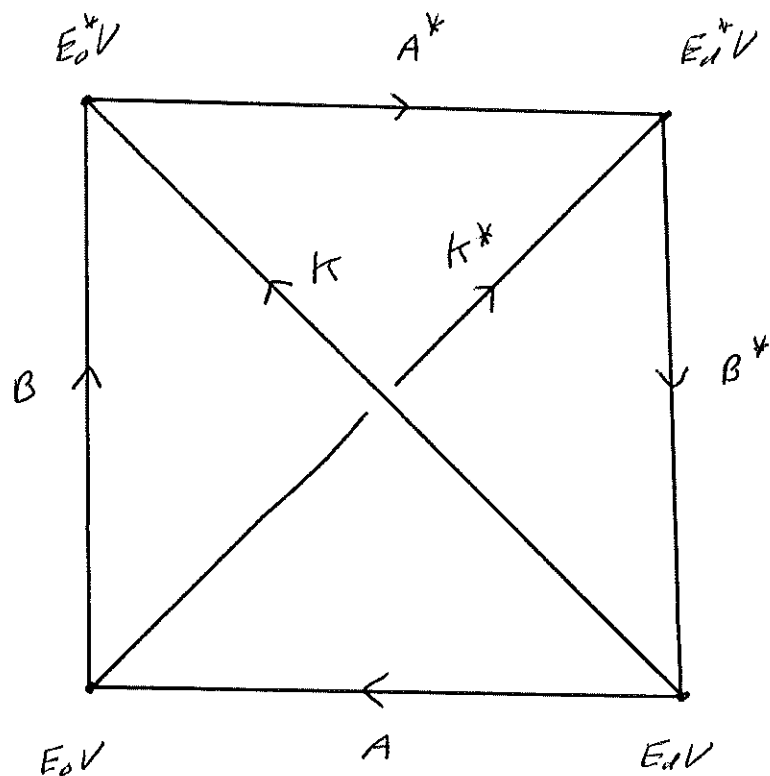
We have



DEF 2 We define

$$B, B^*, K, K^* \in \text{End}(V)$$

as follows:



For example, for $0 \leq i \leq d$

$$(E_0^i V + \dots + E_i^i V) \cap (E_i V + \dots + E_d V)$$

is an eigenspace for K with eigenvalue q^{2i-d} .

Next goal: find the relations satisfied by

$$A, A^*, B, B^*, K, K^*, K^{-1}, K^{*-1}.$$

We will use the following handy facts.

For any $\gamma \in \text{End}(V)$ and $\theta \in F$

define

$$V_\gamma(\theta) = \{v \in V \mid \gamma v = \theta v\}$$

LEM 3 Given $Y, Z \in \text{End}(V)$

Given $0 \neq \theta \in F$

TFAE:

$$(i) \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I \quad \text{on } V_Y(\theta)$$

$$(ii) \quad (Z - \theta^{-1}I) V_Y(\theta) \subseteq V_Y(q^{-2}\theta)$$

pf On $V_Y(\theta)$,

$$\begin{aligned} qYZ - q^{-1}ZY &= (q - q^{-1})I \\ &= q(Y - q^{-2}\theta I)(Z - \theta^{-1}I) \end{aligned}$$

Result follows. □

the eqn in (i) is called a

" q -Weyl relation"

LEM 4 Given $Y, Z \in \text{End}(V)$

Given $\alpha \neq 0 \in \mathbb{F}$ TFAE:

$$(i) \quad \frac{YZY - Y^TZY}{Y - Y^T} = I \quad \text{on } V_Z(\alpha)$$

$$(ii) \quad (Y - \alpha^T I) V_Z(\alpha) \subseteq V_Z(Y^2 \alpha)$$

p f In Lem 3 replace (Y, Z, γ)

by (Z, Y, γ^T) .

□

LEM 5 Given $Y, Z \in \text{End}(V)$
 Given $0 \neq \theta \in F$ TFAE

$$(i) \quad Y^3 Z - [3]_q Y^2 Z Y + [3]_q Y Z Y^2 - Z Y^3 \\ \text{on } V_Y(\theta)$$

$$(ii) \quad Z V_Y(\theta) \subseteq V_Y(q^2 \theta) + V_Y(\theta) + V_Y(q^{-2} \theta)$$

pt We saw this earlier. □

(i) LEM 6 We have

$$(i) \quad \frac{qAB - q^*BA}{q - q^*} = I$$

$$(ii) \quad \frac{qBA^* - q^*A^*B}{q - q^*} = I$$

$$(iii) \quad \frac{qA^*B^* - q^*B^*A^*}{q - q^*} = I$$

$$(iv) \quad \frac{qB^*A - q^*A^*B^*}{q - q^*} = I$$

pf In each case, combine Lem 1 and
Lem 3 / Lem 4

□

LEM 7 We have

$$(i) \quad \frac{qK^*A - q^*AK^*}{q - q^*} = I$$

$$(ii) \quad \frac{qKA^* - q^*A^*K}{q - q^*} = I$$

$$(iii) \quad \frac{qAK^* - q^*K^*A}{q - q^*} = I$$

$$(iv) \quad \frac{qA^*K^{**} - q^*K^{**}A^*}{q - q^*} = I$$

pf In each case, apply Lem 1 and Lem 3 / Lem 4 $\square$

We now give the actions of B, B^* on the decompositions of V from the tetrahedron diagram.

LEM 8 Let $\{W_i\}_{i=0}^d$ denote a decomp of V from the tetrahedron diagram. For $0 \leq i \leq d$ the action of B and B^* on W_i is:

decomp	B -action	B^* -action
	$(B - q^{d-2i} I) W_i \subseteq W_{i-1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i+1}$
	$(B - q^{2i-d} I) W_i \subseteq W_{i+1}$	$(B^* - q^{2i-d} I) W_i \subseteq W_{i-1}$
	$(B - q^{2i-d} I) W_i \subseteq W_{i-1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i+1}$
	$(B - q^{2i-d} I) W_i = 0$	$B^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$
	$(B - q^{2i-d} I) W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i-1}$
	$B W_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(B^* - q^{d-2i} I) W_i = 0$