

Math 846

Lecture 33

Assume $\text{char}(\mathbb{F}) = 0$, $d \in \mathbb{N}$

We continue to discuss the Leonard system Φ
with param array

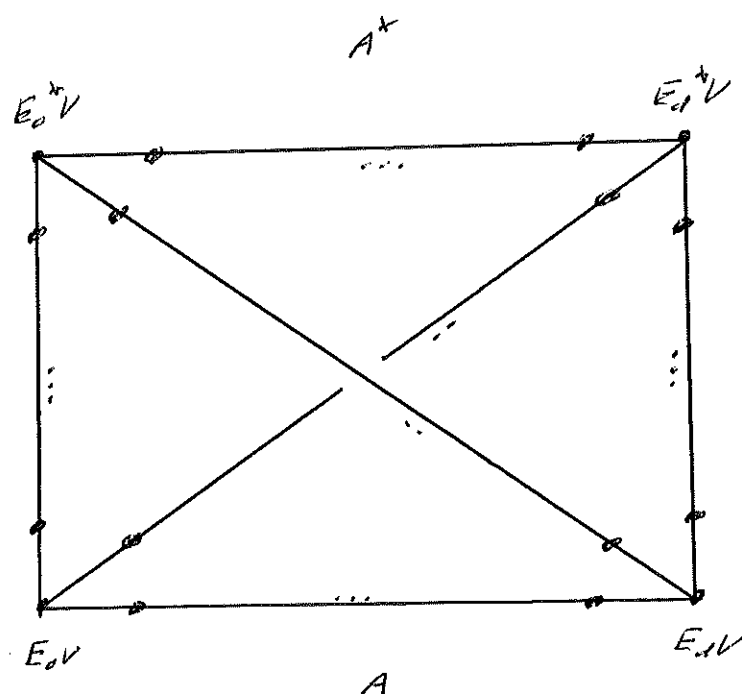
$$\theta_i = d - 2i \quad \theta_i^* = d - 2i \quad (0 \leq i \leq d)$$

$$\varphi_i = -2i(d-i+1) \quad \phi_i = 2i(d-i+1) \quad (1 \leq i \leq d)$$

Recall

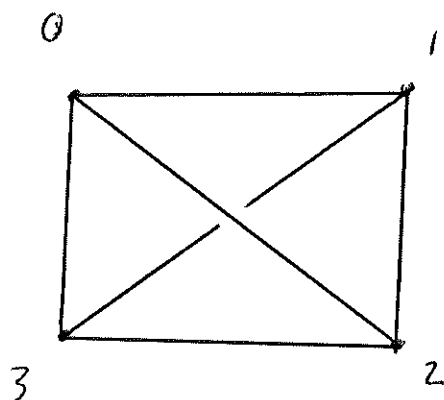
$$[A, [A, A^*]] = 4A^*, \quad [A^*, [A^*, A]] = 4A$$

Recall the tetrahedron diagram for Φ :



The diagram shows 6 (or 12) decompositions of V .

For notational convenience, we label the corners of the diagram:

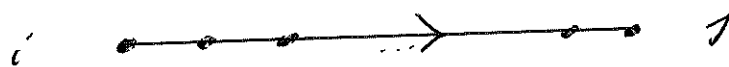


Define the set

$$\Pi = \{0, 1, 2, 3\}$$

For distinct $i, j \in \Pi$ let

$[i, j]$ denote the decomposition of V "from i to j "



For example,

the decomp $\{E_i V\}_{i=0}^d$ is labelled $[3, 2]$

the decomp $\{E_i^V V\}_{i=0}^d$ is labelled $[0, 1]$

Note for distinct $i, j \in \mathbb{I}$

the decomp $[i, j]$ is the inverse of $[j, i]$

For dist $i, j \in \mathbb{I}$ define

$$X_{i,j} \in \text{End}(V)$$

as follows.

Let $\{W_\ell\}_{\ell=0}^d$ denote the decomp $[i, j]$.

For $0 \leq \ell \leq d$,

W_ℓ is an eigenspace for $X_{i,j}$ with equal $2\ell-d$

For example,

$$A = X_{2,3}$$

$$A^* = X_{1,0}$$

Prop 158 For distinct $\lambda, \mu \in \Pi$ we express

X_{ij} in terms of A, A^*

X_{ij}	X_{ij} in terms of A, A^*
X_{01}	$-A^*$
X_{10}	A^*
X_{12}	$A^* - A - \frac{[A, A^*]}{2}$
X_{21}	$A - A^* + \frac{[A, A^*]}{2}$
X_{23}	A
X_{32}	$-A$
X_{30}	$A^* - A + \frac{[A, A^*]}{2}$
X_{03}	$A - A^* - \frac{[A, A^*]}{2}$
X_{02}	$-A - A^* + \frac{[A, A^*]}{2}$
X_{20}	$A + A^* - \frac{[A, A^*]}{2}$
X_{13}	$A + A^* + \frac{[A, A^*]}{2}$
X_{31}	$-A - A^* - \frac{[A, A^*]}{2}$

pf Consider X_{02} .

Let $\{U_i\}_{i=0}^d$ denote the decomp $[0, 2]$ of V .

Pick $0 \neq v \in E_0 V = U_0$

For $0 \leq i \leq d$

$T_i(A)v$ is basis for U_i

rel basis $\{T_i(A)v\}_{i=0}^d$ for V

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ 1 & \theta_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \theta_d \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & & & 0 \\ & \theta_1^* & & \\ & & \ddots & \\ 0 & & & \theta_d^* \end{pmatrix}$$

$$X_{02}: \text{diag}(-d, 2-d, \dots, d)$$

Using these matrix reps, one checks

$$X_{02} = -A - A^* + \frac{[A, A^*]}{2}$$

The other entries are similarly obtained.

□

Another view

Recall A, A^* generate a Lie subalgebra of $\mathfrak{gl}(V)$ iso to $\mathfrak{sl}_2(\mathbb{F})$.

View V as $\mathfrak{sl}_2(\mathbb{F})$ -module on which

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ acts as } A$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ acts as } A^*$$

Then

$\mathfrak{sl}_2$ element	acts on V as
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	X_{01}
$\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$	X_{12}
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	X_{23}
$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$	X_{30}
$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$	X_{02}
$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$	X_{13}

Next goal: Find the relations between

$$X_{ij} \quad i, j \in \mathbb{I} \quad i \neq j.$$

Prop 159 We have

(i) For distinct $i, j \in \mathbb{I}$

$$X_{ij} + X_{ji} = 0$$

(ii) For mutually dist $h, i, j \in \mathbb{I}$

$$[X_{hi}, X_{ij}] = 2X_{hi} + 2X_{ij}$$

(iii) For mut dist $h, i, j, k \in \mathbb{I}$

$$[X_{hi}, [X_{hi}, [X_{hi}, X_{jk}]]] = 4[X_{hi}, X_{jk}]$$

(Jordan/Grady relation)

pf This is checked using the 2×2 matrix reps. □

DEF 160 let $\text{char}(\mathbb{F}) = 0$ the

tetrahedron Lie algebra over $\mathbb{F}$, denoted $\boxtimes$,

is defined by generators

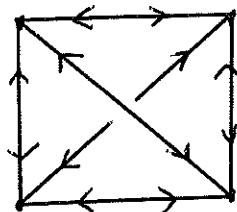
$$X_{i,j} \quad i, j \in \mathbb{I} \quad i \neq j$$

and the relations (i) - (iii) from Prop 159.

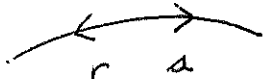
— • —

It turns out $\boxtimes$ is infinite dim'l.

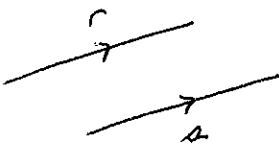
We interpret $\boxtimes$ with a diagram:



• each directed arc represents a generator β $\boxtimes$

•  means $r + a = 0$

•  means $[r, a] = 2r + 2a$

•  means $[r, [r, [r, a]]] = 4[r, a]$
 $[a, [a, [a, r]]] = 4[a, r]$
 (Dolan/Grady relations)

For previous Leonard system Φ on V

V becomes an irred $\boxtimes$ -module on which

each $\boxtimes$ -generator $X_{i,j}$ acts as $x_{i,j}$

— 0 —

The student presentations will describe
how general f.d. irred $\boxtimes$ -modules are
related to tridiagonal pairs.

Ex 161 Assume $\text{char}(\mathbb{F}) = 2$

Let $V =$ vector space over $\mathbb{F}$ with $\dim(V) = 2$

Recall the set

$$\mathbb{I} = \{0, 1, 2, 3\}$$

Let

$$U_i \quad i \in \mathbb{I}$$

denote mutually distinct 1 dim'l subspaces of V .

For distinct $i, j \in \mathbb{I}$ define

$$X_{ij} \in \text{End}(V)$$

as follows. Obs

$$V = U_i + U_j \quad (\text{direct sum}).$$

Require

$$(X_{ij} + I) U_i = 0,$$

$$(X_{ij} - I) U_j = 0$$

Show that the above maps X_{ij} satisfy

the relations Prop 159 (i)-(iii). In other words

V becomes a $\boxtimes$ -module on which $X_{ij} = X_{ji}$ for all $i, j \in \mathbb{I}$, $i \neq j$.

III

Tridiagonal pairs and quantum groups.

Motivation

Assume

 $\mathbb{F}$ arb, $0 \neq q \in \mathbb{F}$ $q^2 \neq 1$

Recall the algebra U_q^+ is defined by two generators that satisfy the q -Serre relations.

We saw that if q is not a root of unity, then on each non nil finite dim'l U_q^+ -module the U_q^+ -generators act as a tridiagonal pair whose eigenvalues / dual-eigenvalues form a q^2 -geometric progression.

Next goal: show how such tridiagonal pairs are related to the q -tetrahedron algebra $\mathbb{T}_q$ and the quantum group $U_q(\hat{\mathfrak{sl}}_2)$.