

Math 846

Lecture 32

Recall

$$\mathbb{F} \text{ arb } d \in \mathbb{N}$$

$$V = \text{vs dim } dH$$

Fix Leonard system on V :

$$\Phi = (A, \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

Recall polynomials p_i, u_i, v_i LEM 151 For $0 \leq i \leq d$

$$p_i(\theta_0) = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\tilde{\tau}_i^*(\theta_i^*)}$$

pf We saw

$$u_i = \frac{p_i}{p_i(\theta_0)}$$

and

$$u_i = \sum_{h=0}^i \frac{\tilde{\tau}_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} \tilde{\tau}_h$$

Compare leading coeff:

$$\frac{1}{p_i(\theta_0)} = \frac{\tilde{\tau}_i^*(\theta_i^*)}{\varphi_1 \cdots \varphi_i}$$

✓

□

Thm 152 For $0 \leq i \leq d$ the polynomial

$$p_i = \sum_{h=0}^i \frac{\varphi_1 \cdots \varphi_i}{\varphi_1 \cdots \varphi_h} \frac{\tau_h^*(\theta_i^*)}{\tau_i^*(\theta_i^*)} \tau_h$$

pf

$$p_i = p_i(\theta_0) u_i$$

$$= \frac{\varphi_1 \cdots \varphi_i}{\tau_i^*(\theta_i^*)} \sum_{h=0}^i \frac{\tau_h^*(\theta_i^*)}{\varphi_1 \cdots \varphi_h} \tau_h$$

□

Thm 153

We have

$$(i) \quad b_i = \varphi_{i:n} \frac{\tau_i^*(\theta_i^*)}{\tau_{i:n}^*(\theta_{i:n}^*)} \quad (0 \leq i \leq d-1)$$

$$(ii) \quad c_i = \phi_i \frac{\gamma_{d-i}^*(\theta_{i:n}^*)}{\gamma_{d-i:n}^*(\theta_{i:n}^*)} \quad (1 \leq i \leq d)$$

pf (i)

We have

$$p_i(\theta_0) = \frac{\varphi_1 \varphi_2 \dots \varphi_i}{\tau_i^*(\theta_i^*)}$$

$$=$$

$$b_0 b_1 \dots b_{i-1}$$

$$b_i = \frac{p_{i:n}(\theta_0)}{p_i(\theta_0)} = \frac{\varphi_1 \dots \varphi_{i:n}}{\tau_{i:n}^*(\theta_{i:n}^*)} \frac{\tau_i^*(\theta_i^*)}{\varphi_1 \dots \varphi_i} \quad \checkmark$$

(ii) By constr

$$c_i = b_{d-i} \quad \downarrow$$

Recall Φ^ψ has param array

$$\left(\{ \theta_i \}_{i=0}^d ; \{ \theta_{d-i}^* \}_{i=0}^d ; \{ \phi_{d-i:n} \}_{i=1}^d ; \{ \varphi_{d-i:n} \}_{i=1}^d \right)$$

Use this and (i)

□

Thm 154

For $1 \leq i \leq d$

$$x_i = \psi_i \phi_i \frac{\tau_{i-1}^*(\theta_{i-1}^*) y_{d-i}^*(\theta_i^*)}{\tau_i^*(\theta_i^*) y_{d-i-1}^*(\theta_{i-1}^*)}$$

pf

Use

$$x_i = b_{i-1} c_i$$

and Thm 153.

□

Thm 155

We have

$$V = \frac{\gamma_d(\theta_0) \gamma_d^*(\theta_0^v)}{\phi_1 \phi_2 \dots \phi_d}$$

pf By Prop 124

$$\underbrace{|\theta_0 - \theta_1| |\theta_0 - \theta_2| \dots |\theta_0 - \theta_d|}_{= \gamma_d(\theta_0)} = V c_1 c_2 \dots c_d$$

By Thm 153 (ii)

$$c_1 c_2 \dots c_d = \phi_1 \phi_2 \dots \phi_d \frac{1}{\gamma_d^*(\theta_0^v)}$$

□

Thm 156 For $0 \leq i \leq d$

$$k_i = \frac{\varphi_1 \varphi_2 \dots \varphi_i}{\phi_1 \phi_2 \dots \phi_i} \frac{\gamma_d^*(\theta_0^*)}{\gamma_i^*(\theta_i^*) \gamma_{d-i}^*(\theta_i^*)}$$

p.f. Use Thm 153 and

$$k_i = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$$

□

Thm 157

For $0 \leq i \leq d$

$$m_i = \frac{\varphi_1 \varphi_2 \dots \varphi_i \quad \phi_1 \phi_2 \dots \phi_{d-i}}{\eta_d^*(\theta_0^*) \quad \gamma_i(\theta_i) \eta_{d-i}(\theta_i)}$$

pf Use

$$m_i = k_i^* \mathcal{V}^{-1}$$

and pms 155, 156

□

Aside on hypergeometric series and basic
hypergeometric series.

For $a \in \mathbb{F}$ and $n \in \mathbb{N}$ define

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1)$$

Interp $(a)_0 = 1$

obs

$$(a)_n = 0 \quad \text{iff} \quad a \in \{0, -1, -2, \dots, 1-n\}$$

let $z = indet$

For $r, s \in \mathbb{N}$ and

$$a_1, a_2, \dots, a_r; \quad b_1, b_2, \dots, b_s \in \mathbb{F}$$

define

$${}_rF_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| z \right) = \sum_{n \in \mathbb{N}} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{z^n}{n!}$$

||

" ${}_rF_s$ hypergeometric series

(1) Series terminates iff at least one of $a_1, a_2, \dots, a_r$ is in $\{0, -1, -2, \dots\}$

We are interested in the terminating case

Fix $0 \neq q \in \mathbb{F}$ st $q^2 \neq 1$

For $a \in \mathbb{F}$ and $n \in \mathbb{N}$ define

$$(a; q)_n = (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1})$$

Interp $(a; q)_0 = 1$

Obs $(a; q)_n = 0$ iff $a \in \{1, q, q^2, \dots, q^{1-n}\}$

For $r, a \in \mathbb{N}$ and

$$a_1, a_2, \dots, a_r \quad b_1, b_2, \dots, b_s \in \overline{\mathbb{F}}$$

define

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} / q; z \right)$$

$$= \sum_{n \in \mathbb{N}} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n} (-1)^{(1+a-r)n} q^{\binom{1+a-r}{2} n}$$

" ${}_r\phi_s$ basic hypergeometric series "

Series terminates iff at least one of

$$a_1, a_2, \dots, a_r \text{ is in } \{1, q^{-1}, q^{-2}, \dots\}$$

We are interested in terminating case.

A series ${}_rF_s$ or ${}_r\phi_s$ is balanced whenever

$$r = s + 1$$

We are interested in the balanced case.

Example

Recall the Leonard systems

attached to the hypercubes:

$$d \in \mathbb{N}$$

$$\text{char}(\mathbb{F}) = d,$$

$$c_i = c_i^* = 1$$

$$a_i = a_i^* = 0$$

$$(0 \leq i \leq d)$$

$$b_i = b_i^* = d - i$$

$$\theta_i = \theta_i^* = d - 2i$$

Find parameter array.

Find φ_i ($1 \leq i \leq d$)

Recall

$$b_i = \varphi_{i-1} \frac{\gamma_i^* / \theta_i^*}{\gamma_{i-1}^* / \theta_{i-1}^*} \quad (0 \leq i \leq d)$$

So for $1 \leq i \leq d$

$$\varphi_i = \underbrace{b_{i-1}}_{d-i+1} \frac{\gamma_i^* / \theta_i^*}{\gamma_{i-1}^* / \theta_{i-1}^*} \} = (-2)^{i-1} i!$$

$$= -2 i (d-i+1)$$

Find ϕ_i ($1 \leq i \leq d$)

Recall

$$c_i = \phi_i \frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i}^*(\theta_{i-1}^*)} \quad (1 \leq i \leq d)$$

So

$$\phi_i = \frac{c_i}{\prod_{j=1}^i \frac{\eta_{d-j}^*(\theta_j^*)}{\eta_{d-j}^*(\theta_{j-1}^*)}} = \frac{c_i}{\frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i}^*(\theta_{i-1}^*)}} = \frac{c_i \eta_{d-i}^*(\theta_{i-1}^*)}{\eta_{d-i}^*(\theta_i^*)}$$

$$= 2^i (d-i)!$$

param array is

$$\left(\{d-2i\}_{i=0}^d ; \{d-2i\}_{i=0}^d ; \{-2i(d-i)\}_{i=1}^d ; \{2i(d-i)\}_{i=1}^d \right)$$

Find k_i ($0 \leq i \leq d$)

$$k_i = \frac{b_0 b_1 \dots b_i}{c_1 c_2 \dots c_i}$$

$$= \binom{d}{i}$$

Find $\mathcal{V}$

$$\mathcal{V} = \frac{\overbrace{\gamma_d(\theta_0)}^{2^d d!} \overbrace{\gamma_d^*(\theta_0^*)}^{2^d d!}}{\phi_1 \phi_2 \dots \phi_d} \} = 2^d d! d!$$

$$= 2^d$$

Find m_i ($0 \leq i \leq d$)

$$m_i = k_i^* \mathcal{V}^{-1}$$

$$= \binom{d}{i} 2^{-d}$$

Find $u_i(\theta_2)$ ($0 \leq i \leq d$)

$$u_i(\theta_2) = \sum_{h=0}^i \frac{\overbrace{T_h^*(\theta_1^*)}^{= 2^h (-i)_h} \overbrace{T_h(\theta_2)}^{= 2^h (-j)_h}}{\underbrace{\varphi_1 \varphi_2 \dots \varphi_h}_{= 2^h h! (-d)_h}}$$

$$= \sum_{h=0}^i \xrightarrow{d \rightarrow \infty} \frac{(-i)_h (-j)_h}{(-d)_h} \frac{2^h}{h!}$$

$$= {}_2F_0 \left(\begin{matrix} -i & -j \\ & -d \end{matrix} / 2 \right)$$

— 0 —

Every Leonard system can be treated
in a similar manner. See the
Appendix below.

Theorem 34.14 [51, Section 10] Assume $\mathbb{K}$ is algebraically closed. Let q denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a tridiagonal pair on V that has q -geometric type. Then there exists an irreducible $\boxtimes_q$ -module structure on V such that A acts as a scalar multiple of x_{01} and A^* acts as a scalar multiple of x_{23} . Conversely, let V denote a finite dimensional irreducible $\boxtimes_q$ -module. Then the generators x_{01}, x_{23} act on V as a tridiagonal pair of q -geometric type.

We end this section with a conjecture.

Conjecture 34.15 Assume $\mathbb{K}$ is algebraically closed. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V . To avoid degenerate situations we assume q is not a root of unity, where $\beta = q^2 + q^{-2}$, and where β is from Theorem 34.8. Then referring to Definition 34.12, there exists an irreducible $\boxtimes_q$ -module structure on V such that A acts as a linear combination of x_{01}, x_{12}, I and A^* acts as a linear combination of x_{23}, x_{30}, I .

35 Appendix: List of parameter arrays

In this section we display all the parameter arrays over $\mathbb{K}$. We will use the following notation.

Definition 35.1 Let $p = (\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote a parameter array over $\mathbb{K}$. For $0 \leq i \leq d$ we let u_i denote the following polynomial in $\mathbb{K}[\lambda]$.

$$u_i = \sum_{n=0}^i \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}. \quad (134)$$

We call $u_0, u_1, \dots, u_d$ the polynomials that correspond to p .

We now display all the parameter arrays over $\mathbb{K}$. For each displayed array $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ we present $u_i(\theta_j)$ for $0 \leq i, j \leq d$, where $u_0, u_1, \dots, u_d$ are the corresponding polynomials. Our presentation is organized as follows. In each of Example 35.2–35.14 below we give a family of parameter arrays over $\mathbb{K}$. In Theorem 35.15 we show every parameter array over $\mathbb{K}$ is contained in at least one of these families.

In each of Example 35.2–35.14 below the following implicit assumptions apply: d denotes a nonnegative integer, the scalars $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ are contained in $\mathbb{K}$, and the scalars $q, h, h^* \dots$ are contained in the algebraic closure of $\mathbb{K}$.

Example 35.2 (q -Racah) Assume

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \quad (135)$$

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i} \quad (136)$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i), \quad (137)$$

$$\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \quad (138)$$

for $1 \leq i \leq d$. Assume $h, h^*, q, s, s^*, r_1, r_2$ are nonzero and $r_1 r_2 = s s^* q^{d+1}$. Assume none of $q^i, r_1 q^i, r_2 q^i, s^* q^i / r_1, s^* q^i / r_2$ is equal to 1 for $1 \leq i \leq d$ and that neither of $s q^i, s^* q^i$ is equal to 1 for $2 \leq i \leq 2d$. Then $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_4\phi_3 \left(\begin{matrix} q^{-i}, s^* q^{i+1}, q^{-j}, s q^{j+1} \\ r_1 q, r_2 q, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These u_i are the q -Racah polynomials.

Example 35.3 (q -Hahn) Assume

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= h h^* q^{1-2i} (1 - q^i) (1 - q^{i-d-1}) (1 - r q^i), \\ \phi_i &= -h h^* q^{1-i} (1 - q^i) (1 - q^{i-d-1}) (r - s^* q^i) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s^*, r are nonzero. Assume none of $q^i, r q^i, s^* q^i / r$ is equal to 1 for $1 \leq i \leq d$ and that $s^* q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2 \left(\begin{matrix} q^{-i}, s^* q^{i+1}, q^{-j} \\ r q, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These u_i are the q -Hahn polynomials.

Example 35.4 (Dual q -Hahn) Assume

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)(1 - s q^{i+1})q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= h h^* q^{1-2i} (1 - q^i) (1 - q^{i-d-1}) (1 - r q^i), \\ \phi_i &= h h^* q^{d+2-2i} (1 - q^i) (1 - q^{i-d-1}) (s - r q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r, s are nonzero. Assume none of $q^i, r q^i, s q^i / r$ is equal to 1 for $1 \leq i \leq d$ and that $s q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2 \left(\begin{matrix} q^{-i}, q^{-j}, s q^{j+1} \\ r q, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These u_i are the dual q -Hahn polynomials.

Example 35.5 (Quantum q -Krawtchouk) Assume

$$\begin{aligned}\theta_i &= \theta_0 - sq(1 - q^i), \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= -rh^*q^{1-i}(1 - q^i)(1 - q^{i-d-1}), \\ \phi_i &= h^*q^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})(s - rq^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h^*, q, r, s are nonzero. Assume neither of $q^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_2\phi_1\left(\begin{matrix} q^{-i}, q^{-j} \\ q^{-d} \end{matrix} \middle| q, sr^{-1}q^{j+1}\right)$$

for $0 \leq i, j \leq d$. These u_i are the quantum q -Krawtchouk polynomials.

Example 35.6 (q -Krawtchouk) Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1 - q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1}), \\ \phi_i &= hh^*s^*q(1 - q^i)(1 - q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s^* are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and that $s^*q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, s^*q^{i+1}, q^{-j} \\ 0, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These u_i are the q -Krawtchouk polynomials.

Example 35.7 (Affine q -Krawtchouk) Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1 - q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - rq^i), \\ \phi_i &= -hh^*rq^{1-i}(1 - q^i)(1 - q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r are nonzero. Assume neither of q^i, rq^i is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2 \left(\begin{matrix} q^{-i}, 0, q^{-j} \\ rq, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These u_i are the affine q -Krawtchouk polynomials.

Example 35.8 (Dual q -Krawtchouk) Assume

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1}), \\ \phi_i &= hh^*sq^{d+2-2i}(1 - q^i)(1 - q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2 \left(\begin{matrix} q^{-i}, q^{-j}, sq^{j+1} \\ 0, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These u_i are the dual q -Krawtchouk polynomials.

Example 35.9 (Racah) Assume

$$\theta_i = \theta_0 + hi(i + 1 + s), \quad (139)$$

$$\theta_i^* = \theta_0^* + h^*i(i + 1 + s^*) \quad (140)$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*i(i - d - 1)(i + r_1)(i + r_2), \quad (141)$$

$$\phi_i = hh^*i(i - d - 1)(i + s^* - r_1)(i + s^* - r_2) \quad (142)$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = s + s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d . Assume none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of s, s^* is equal to $-i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_4F_3 \left(\begin{matrix} -i, i + 1 + s^*, -j, j + 1 + s \\ r_1 + 1, r_2 + 1, -d \end{matrix} \middle| 1 \right)$$

for $0 \leq i, j \leq d$. These u_i are the Racah polynomials.

Example 35.10 (Hahn) Assume

$$\begin{aligned}\theta_i &= \theta_0 + si, \\ \theta_i^* &= \theta_0^* + h^*i(i+1+s^*)\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= h^*si(i-d-1)(i+r), \\ \phi_i &= -h^*si(i-d-1)(i+s^*-r)\end{aligned}$$

for $1 \leq i \leq d$. Assume h^*, s are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d . Assume neither of $r, s^* - r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s^* \neq -i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3F_2\left(\begin{matrix} -i, i+1+s^*, -j \\ r+1, -d \end{matrix} \middle| 1\right)$$

for $0 \leq i, j \leq d$. These u_i are the Hahn polynomials.

Example 35.11 (Dual Hahn) Assume

$$\begin{aligned}\theta_i &= \theta_0 + hi(i+1+s), \\ \theta_i^* &= \theta_0^* + s^*i\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hs^*i(i-d-1)(i+r), \\ \phi_i &= hs^*i(i-d-1)(i+r-s-d-1)\end{aligned}$$

for $1 \leq i \leq d$. Assume h, s^* are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d . Assume neither of $r, s - r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s \neq -i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3F_2\left(\begin{matrix} -i, -j, j+1+s \\ r+1, -d \end{matrix} \middle| 1\right)$$

for $0 \leq i, j \leq d$. These u_i are the dual Hahn polynomials.

Example 35.12 (Krawtchouk) Assume

$$\begin{aligned}\theta_i &= \theta_0 + si, \\ \theta_i^* &= \theta_0^* + s^*i\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= ri(i-d-1) \\ \phi_i &= (r - ss^*)i(i-d-1)\end{aligned}$$

for $1 \leq i \leq d$. Assume r, s, s^* are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d . Assume $r \neq ss^*$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_2F_1 \left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| r^{-1}ss^* \right)$$

for $0 \leq i, j \leq d$. These u_i are the Krawtchouk polynomials.

Example 35.13 (Bannai/Ito) Assume

$$\theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i), \quad (143)$$

$$\theta_i^* = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i) \quad (144)$$

for $0 \leq i \leq d$ and

$$\varphi_i = \begin{cases} -4hh^*i(i + r_1), & \text{if } i \text{ even, } d \text{ even;} \\ -4hh^*(i - d - 1)(i + r_2), & \text{if } i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & \text{if } i \text{ even, } d \text{ odd;} \\ -4hh^*(i + r_1)(i + r_2), & \text{if } i \text{ odd, } d \text{ odd,} \end{cases} \quad (145)$$

$$\phi_i = \begin{cases} 4hh^*i(i - s^* - r_1), & \text{if } i \text{ even, } d \text{ even;} \\ 4hh^*(i - d - 1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & \text{if } i \text{ even, } d \text{ odd;} \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ odd} \end{cases} \quad (146)$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = -s - s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is either 0 or an odd prime greater than $d/2$. Assume neither of $r_1, -s^* - r_1$ is equal to $-i$ for $1 \leq i \leq d, d - i$ even. Assume neither of $r_2, -s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d, i$ odd. Assume neither of s, s^* is equal to $2i$ for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. We call the corresponding polynomials from Definition 35.1 the Bannai/Ito polynomials [11, p. 260].

Example 35.14 (Orphan) For this example assume $\mathbb{K}$ has characteristic 2. For notational convenience we define some scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $\mathbb{K}$. We define $\gamma_i = 0$ for $i \in \{0, 3\}$ and $\gamma_i = 1$ for $i \in \{1, 2\}$. Assume

$$\theta_i = \theta_0 + h(si + \gamma_i), \quad (147)$$

$$\theta_i^* = \theta_0^* + h^*(s^*i + \gamma_i) \quad (148)$$

for $0 \leq i \leq 3$. Assume $\varphi_1 = hh^*r, \varphi_2 = hh^*, \varphi_3 = hh^*(r + s + s^*)$ and $\phi_1 = hh^*(r + s(1 + s^*)), \phi_2 = hh^*, \phi_3 = hh^*(r + s^*(1 + s))$. Assume each of h, h^*, s, s^*, r is nonzero. Assume neither of s, s^* is equal to 1 and that r is equal to none of $s + s^*, s(1 + s^*), s^*(1 + s)$. Then the sequence $(\theta_i, \theta_i^*, i = 0..3; \varphi_j, \phi_j, j = 1..3)$ is a parameter array over $\mathbb{K}$ which has diameter 3. We call the corresponding polynomials from Definition 35.1 the orphan polynomials.

Theorem 35.15 Every parameter array over $\mathbb{K}$ is listed in at least one of the Examples 35.2–35.14.