

Math 846

Lecture 31

Recall

 $\mathbb{F}$ arb, $d \in \mathbb{N}$ $V =$ vector space $\dim d+1$ Fix Leonard system in V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Recall polynomials u_i, v_i, p_i

Next goal: explain the "orthogonality" of
these polynomials. What is orthog to what?

Thm 14.5 We give the orthogonality for $\{v_i\}_{i=0}^d$:(i) for $0 \leq i, j \leq d$

$$\sum_{r=0}^d v_i(\theta_r) v_j(\theta_r) k_r^* = \delta_{ij} \nu k_i$$

(ii) for $0 \leq i, z \leq d$

$$\sum_{i=0}^d v_i(\theta_r) v_i(\theta_z) k_i^{-1} = \delta_{r,z} \nu k_r^{*-1}$$

[Above are analogous to the row and column orthogonality
for the character table of a finite group]

pf (:) Pick

$$u \neq v \in E_0 V,$$

$$u \neq v \in E_0^* V.$$

Eval

$$\langle E_i^* u, E_j^* u \rangle$$

in 2 ways.

By thm 137,

$$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i v^{-1} \|u\|^2$$

By thm 144,

$$\langle E_i^* u, E_j^* u \rangle = \frac{\langle u, v \rangle^2}{\|v\|^2 \|v\|^2} \sum_{r=0}^d \sum_{s=0}^d v_i(\theta_r) v_j(\theta_s) \underbrace{\langle E_r v, E_s v \rangle}_{//}$$

$$\delta_{rs} k_r v^{-1} \|v\|^2$$

$$= \frac{\langle u, v \rangle^2}{\|v\|^2} \sum_{r=0}^d v_i(\theta_r) v_j(\theta_r) k_r^* v^{-1}$$

So

$$\sum_{r=0}^d v_i(\theta_r) v_j(\theta_r) k_r^* = \delta_{ij} k_i \frac{\|u\|^2 \|v\|^2}{\langle u, v \rangle^2}$$

$$= \delta_{ij} k_i v.$$

(ii) Apply (i) to $\mathcal{F}^*$ and use

Askey-Wilson duality:

$$\sum_{i=0}^d \underbrace{v_r^*(\theta_i^v)}_{// \text{AND}} \underbrace{v_a^*(\theta_i^*)}_{|| \text{AND}} k_i = \delta_{r,a} \vee k_r^*$$

$$v_i(\theta_r) \frac{k_r^*}{k_i} \quad v_i(\theta_a) \frac{k_a^*}{k_i}$$

Result follows.

□

Thm 146 We give the orthog rels for $\{u_i\}_{i=0}^d$

(i) for $0 \leq i, j \leq d$

$$\sum_{r=0}^d u_i(\theta_r) u_j(\theta_r) k_r^* = \delta_{ij} \vee k_i^{*-1}$$

(ii) for $0 \leq r, a \leq d$

$$\sum_{i=0}^d u_i(\theta_r) u_i(\theta_a) k_i = \delta_{r,a} \vee k_r^{*-1}$$

pf Eval thm 145 using

$$v_l = u_l k_l \quad (0 \leq l \leq d)$$

□

Thm 147 We give the orthog rels for $\{p_i\}_{i=0}^d$

(i) for $0 \leq i, j \leq d$

$$\sum_{r=0}^d p_i(a_r) p_j(a_r) m_r = \delta_{ij} x_1 x_2 \cdots x_d$$

(ii) for $0 \leq r, s \leq d$

$$\sum_{i=0}^d \frac{p_i(a_r) p_i(a_s)}{x_1 x_2 \cdots x_d} = \delta_{r,s} m_r^{-1}$$

pf Eval thm 145 using

$$V_L = \frac{p_L}{c_1 \cdots c_d}$$

$$k_L^* = m_L \quad \checkmark$$

$$k_L = \frac{b_0 b_1 \cdots b_{d-1}}{c_1 c_2 \cdots c_d}$$

$$x_L = c_d b_{d-1}$$

□

DEF 148 We define

$$P \in \text{Mat}_{dn}(\mathbb{F})$$

with (i,i) -entry

$$P_{i,i} = v_j(\theta_i) \quad (0 \leq i, j \leq d)$$

To clarify P , consider col 0 and row 0.

col 0:

$$P_{i,0} = v_0(\theta_i) = 1 \quad (0 \leq i \leq d)$$

row 0:

$$P_{0,j} = v_j(\theta_0) = k_j \quad (0 \leq j \leq d)$$

$$P: \begin{pmatrix} k_0=1, & k_1 & k_2 & \dots & k_d \\ 1 & & & & \\ 1 & & & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix}$$

Thm 149 We have

$$P^* P = P P^* = \nu I$$

p.f. Suf to show $P^* P = \nu I$

For $0 \leq i, j \leq d$ check

$$(P^* P)_{ij} \stackrel{?}{=} \delta_{ij} \nu$$

//

$$\sum_{r=0}^d (P^* /_{ir} P)_{rj}$$

//

$$\sum_{r=0}^d v_r^*(a_i^*) v_j(a_r)$$

// two

$$v_i(a_r) \frac{k_r^*}{k_i}$$

$$= k_i^{-1} \sum_{r=0}^d v_i(a_r) v_j(a_r) k_r^*$$

$$\underbrace{\hspace{10em}}_{\delta_{ij} \nu k_i}$$

ok.

□

Another view of Thm 149:

L3/p8

Pick

$$0 \neq u \in E_0 V$$

$$0 \neq v \in E_0^* V$$

Φ -standard basis for V :

$$E_i^* u$$

oried

(*)

Φ^* -standard basis for V

$$E_i v$$

oried

(**)

By Thm 144

the transition matrix from (**) to (*) is $\frac{\langle u, v \rangle}{\|v\|^2} P$

, , (*) to (**) is $\frac{\langle u, v \rangle}{\|u\|^2} P^*$

So

$$I = \underbrace{\frac{\langle u, v \rangle^2}{\|u\|^2 \|v\|^2}}_{\|v^{-1}\|^2} P P^*$$

Recall for $X \in \text{End}(V)$, matrix X^b represents X wrt (*). Let $X^\#$ represent X wrt (**).

By linearity

$$X^\# P = P X^b$$

Next goal:

Recall the polynomials p_i, u_i, v_i

We defined these polys using a 3-term recurrence.

We now express these polynomials in closed form.

Recall the param array of Φ :

$$\left(\{ \theta_i \}_{i=0}^d ; \{ \theta_i^* \}_{i=0}^d ; \{ \psi_i \}_{i=1}^d ; \{ \phi_i \}_{i=1}^d \right)$$

Recall the polynomials

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \quad (0 \leq i \leq d+1)$$

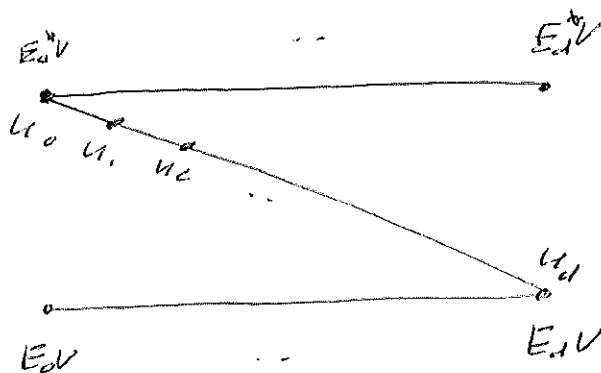
$$\gamma_i = (\lambda - \theta_1)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}),$$

We will use a split decomp of V :

$$V = \sum_{i=0}^d U_i \quad (\text{dir sum})$$

where

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)$$



Pick $0 \neq v \in E_0^* V$.

Recall $T_i(A)v$ is a basis for U_i ($0 \leq i \leq d$)

Rel basis $\{T_i(A)v\}_{i=0}^d$ for V ,

$$A: \begin{pmatrix} \theta_0^* & & & 0 \\ 1 & \theta_1^* & & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 & \theta_d^* \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* \varphi_1 & & & 0 \\ \theta_1^* \varphi_2 & & & \\ & \ddots & & \\ 0 & & & \varphi_d \\ & & & \theta_d^* \end{pmatrix}$$

Thm 150 For $0 \leq i \leq d$ the polynomial

$$u_i = \sum_{h=0}^i \frac{T_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} T_h$$

pf u_i has degree i

Write

$$u_i = \sum_{h=0}^i \alpha_h T_h \quad \alpha_h \in \mathbb{F}$$

show

$$\alpha_h = \frac{T_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} \quad 0 \leq h \leq i$$

so f to show

$$\alpha_0 = 1$$

$$\alpha_h = \alpha_{h-1} \frac{\theta_i^* - \theta_{h-1}^*}{\varphi_h} \quad 1 \leq h \leq i \quad (*)$$

(**)

pf (*):

$$u_i(\theta_0) = \sum_{h=0}^i \alpha_h \underbrace{T_h(\theta_0)}_{\begin{cases} 0 & \text{if } h \geq 1 \\ 1 & \text{if } h = 0 \end{cases}}$$

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$$\text{So } \alpha_0 = 1 \checkmark$$

pf (**):

$$V_i(A) E_0^* V = E_i^* V$$

so

$$u_i(A) E_0^* V = E_i^* V$$

$$u_i = \frac{v_i}{k_i}$$

so

$$(A^* - \theta_i^* I) u_i(A) v = 0$$

obs

$$\begin{aligned} 0 &= (A^* - \theta_i^* I) u_i(A) v \\ &= (A^* - \theta_i^* I) \sum_{h=0}^i \alpha_h T_h(A) v \\ &= \sum_{h=0}^i \alpha_h \left(A^* - \theta_h^* I + (\theta_h^* - \theta_i^*) I \right) T_h(A) v \\ &= \sum_{h=0}^i \alpha_h \left(\varphi_h T_{h+1}(A) v + (\theta_h^* - \theta_i^*) T_h(A) v \right) \\ &= \sum_{h=0}^{i-1} T_h(A) v \left(\underbrace{\alpha_{h+1} \varphi_{h+1} + \alpha_h (\theta_h^* - \theta_i^*)}_{\text{must be 0}} \right) \end{aligned}$$

so

$$\alpha_h = \alpha_{h+1} \frac{\theta_i^* - \theta_{h+1}^*}{\varphi_h} \quad 1 \leq h \leq i$$

OK.

□