

Math 846

Lecture 30

Recall

$\mathbb{F}$ arb $d \in \mathbb{N}$

$V = \text{vector space dim } d$

Fix LS on V :

$$\mathbb{E} = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Recall the polynomials

$$v_i = \frac{p_i}{c_1 c_2 \cdots c_i}$$

$$u_i = \frac{p_i}{p_i(0)} = \frac{p_i}{b_0 b_1 \cdots b_{i-1}} \quad 0 \leq i \leq d$$

LEM 135

Fa as is d

$$v_i = k_i u_i$$

pf

$$\frac{p_i}{c_1 u_1 \dots c_i} \stackrel{?}{=} k_i \frac{p_i}{p_i(0) \leftarrow b_1 \dots b_i}$$

$\frac{b_1 \dots b_i}{c_1 \dots c_i}$

✓

ok

□

Next we introduce a bilinear form $\langle, \rangle$ on V .

Caution Here $\mathbb{F}$ is arb. We are not

discussing a Hermitean form or Euclidean form.

Aside on linear algebra

A bilinear form on V is a map

$$\langle, \rangle: V \times V \rightarrow F$$

such that for $u, v, w \in V$ and $\alpha \in F$,

- $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
- $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
- $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$

A bilinear form $\langle, \rangle$ is symmetric whenever

$$\langle u, v \rangle = \langle v, u \rangle \quad \forall u, v \in V$$

For any bilinear form $\langle, \rangle$ on V TFAE:

(i) $\exists 0 \neq u \in V$ st $\langle u, v \rangle = 0 \quad \forall v \in V$

(ii) $\exists 0 \neq v \in V$ st $\langle u, v \rangle = 0 \quad \forall u \in V$

call $\langle, \rangle$ degenerate if (i), (ii) hold and nondegenerate otherwise.

Ex 136 let X denote any element of $\text{End}(V)$. Then $\exists$ a bilinear form $\langle \cdot, \cdot \rangle$ on V st

$$\langle Xu, v \rangle = \langle u, X^*v \rangle$$

for all $u, v \in V$ and $X \in \text{End}(V)$.

The form $\langle \cdot, \cdot \rangle$ is unique up to mult by a nonzero scalar in $\mathbb{F}$. The form $\langle \cdot, \cdot \rangle$ is nondegenerate.

The form $\langle \cdot, \cdot \rangle$ is not symmetric in general.

— v —

We return our attention to Leonard system Φ

Recall $\exists$ anti-invariant t of $\text{End}(V)$ that fixes each of A, A^* .

From now on, $\langle, \rangle$ denotes the corresp bilinear form on V from EX 136.

By construction,

$$\langle Xu, v \rangle = \langle u, X^t v \rangle$$

for all $u, v \in V$ and $X \in \text{End}(V)$.

Recall

$$\mathcal{D} = \text{subalgebra of } \text{End}(V) \text{ gen by } A, A^*$$

$$\mathcal{D}^* = \dots$$

The anti-invariant t fixes everything in $\mathcal{D}$ and $\mathcal{D}^*$

So for $X \in \mathcal{D} \cup \mathcal{D}^*$,

$$\langle Xu, v \rangle = \langle u, Xv \rangle \quad u, v \in V.$$

We abbrev

$$\|u\|^2 = \langle u, u \rangle \quad \forall u \in V$$

Caution: For $0 \neq u \in V$ might have $\|u\|^2 = 0$

Fix $u \neq 0 \in E_0 V$ and recall the $\mathbb{F}$ -standard basis for V :

$$E_0 u, E_1 u, \dots, E_d u$$

Thm 13.7 For $0 \leq i, j \leq d$,

$$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i \nu^{-1} \|u\|^2$$

p.f. $u = E_0 u$ so

$$\begin{aligned} \langle E_i^* u, E_j^* u \rangle &= \langle E_i^* E_0 u, E_j^* E_0 u \rangle \\ &= \langle u, E_0 E_i^* E_j^* E_0 u \rangle \\ &= \delta_{ij} \langle u, \underbrace{E_0 E_i^* E_0 u}_{\substack{\parallel \\ m_i^* E_0 \\ \parallel \\ k_i \nu^{-1} E_0}} \rangle \end{aligned}$$

$$= \delta_{ij} k_i \nu^{-1} \langle u, E_0 u \rangle$$

$= u$

$$= \delta_{ij} k_i \nu^{-1} \|u\|^2$$

□

Cor 13.8 The bilinear form $\langle \cdot, \cdot \rangle$
is symmetric.

p.f. Refer to Thm 13.7:

$$\langle E_i^* u, E_j^* u \rangle = \langle E_j^* u, E_i^* u \rangle \quad (0 \leq i, j \leq d)$$

□

Next goal:

Pick

$$0 \neq u \in E_0 V$$

$$0 \neq v \in E_0^* V$$

Φ - standard basis for V :

$$E_i^* u \quad 0 \leq i \leq d$$

(*)

Φ^* - standard basis for V^* :

$$E_i v \quad 0 \leq i \leq d$$

(**)

How are (*), (**) related?

We consider inner products and transition matrices

We will also get Askey-Wilson duality.

LEM 139 Pick

$$0 \neq u \in E_0 V,$$

$$0 \neq v \in E_0^* V.$$

Then

(i) Each of $\|u\|^2$, $\|v\|^2$, $\langle u, v \rangle$ is nmo

$$(ii) E_0^* u = \frac{\langle u, v \rangle}{\|v\|^2} v$$

$$(iii) E_0 v = \frac{\langle u, v \rangle}{\|u\|^2} u$$

$$(iv) \vee \langle u, v \rangle^2 = \|u\|^2 \|v\|^2$$

pf (i) $\|u\|^2 \neq 0$ by thm 137 and since $\langle, \rangle$ is nmo

$$\text{sim } \|v\|^2 \neq 0$$

shw $\langle u, v \rangle \neq 0$:

$$\text{Recall } E_0^* E_0 \neq 0$$

so $\exists 0 \neq \alpha \in \mathbb{F}$ st

$$E_0^* u = \alpha v$$

Obs

$$\langle u, v \rangle = \langle u, E_0^* v \rangle = \langle E_0^* u, v \rangle = \alpha \langle v, v \rangle = \alpha \|v\|^2 \neq 0$$

$\begin{matrix} \neq & \neq \\ 0 & 0 \end{matrix}$

(iii) In pf ϕ (i)

$$E_o^x u = v$$

$$\langle u, v \rangle = \alpha \|v\|^2$$

$$\text{So } \alpha = \frac{\langle u, v \rangle}{\|v\|^2}$$

(iii) Similar to (ii)

$$(iv) \text{ Using } \forall E_o E_o^x E_o = E_o,$$

$$\begin{aligned} \langle u, v \rangle &= \langle E_o u, v \rangle \\ &= v \langle E_o E_o^x E_o u, v \rangle \\ &\quad \quad \quad \underbrace{\quad}_{=u} \end{aligned}$$

$$= v \langle E_o E_o^x u, v \rangle$$

$$= v \langle \underbrace{E_o^x u}_u, \underbrace{E_o v}_v \rangle$$

$$\frac{\langle u, v \rangle}{\|v\|^2} v \quad \frac{\langle u, v \rangle}{\|u\|^2} u$$

$$= \frac{\langle u, v \rangle^2}{\|u\|^2 \|v\|^2} \underbrace{\langle v, u \rangle}_{= \langle u, v \rangle}$$

Result follows since $\langle u, v \rangle \neq 0$

□

Prop 140 Pick

$$0 \neq u \in E_0 V_i$$

$$0 \neq v \in E_0^* V$$

Then for $0 \leq i, j \leq d$

$$\langle E_i^* u, E_j v \rangle = v^{-1} k_i k_j^* u_i(\theta_j) \langle u, v \rangle$$

$$\begin{aligned}
 \text{pf } \langle E_i^* u, E_j v \rangle &= \langle v_i(A) E_0^* u, E_j v \rangle \\
 &= \langle E_0^* u, v_i(A) E_j v \rangle \\
 &= v_i(\theta_j) \langle E_0^* u, E_j v \rangle \\
 &= v_i(\theta_j) \langle E_0^* u, v_j^*(A^*) E_0 v \rangle \\
 &= v_i(\theta_j) \langle v_j^*(A^*) E_0^* u, E_0 v \rangle \\
 &= \underbrace{v_i(\theta_j)}_{//} \underbrace{v_j^*(\theta_0^*)}_{//} \underbrace{\langle E_0^* u, E_0 v \rangle}_{\frac{\langle u, v \rangle}{\|v\|^2} v} \\
 &= u_i(\theta_j) k_i \underbrace{k_j^* \frac{\langle u, v \rangle^2}{\|u\|^2 \|v\|^2}}_{//} \langle u, v \rangle \\
 &= u_i(\theta_j) k_i k_j^* \underbrace{\frac{\langle u, v \rangle^2}{\|u\|^2 \|v\|^2}}_{v^{-1}} \langle u, v \rangle
 \end{aligned}$$

□

(1) Thm 141 (Askey-Wilson duality)

For $0 \leq i, j \leq d$

$$u_i(\theta_j) = u_j^*(\theta_i^*)$$

pf We refer to Prop 140:

$$\langle E_i^* u, E_j v \rangle = v^T k_i k_j^* u_i(\theta_j) \langle u, v \rangle$$

Apply Prop 140 to Φ^* to get

$$\langle E_i^* u, E_j v \rangle = v^T k_i k_i^* u_j^*(\theta_i^*) \langle u, v \rangle$$

Result follows since $\langle u, v \rangle \neq 0$.

□

Thm 142 (Askey - Wilson duality, version II)

For $0 \leq i, j \leq d$

$$\frac{v_i(\theta_j)}{k_i} = \frac{v_j^*(\theta_i^*)}{k_j^*}$$

p.f. By Lem 135 and Thm 141 □

Thm 143 (Askey - Wilson duality, version III)

For $0 \leq i, j \leq d$

$$\frac{p_i(\theta_j)}{p_i(\theta_0)} = \frac{p_j^*(\theta_i^*)}{p_j^*(\theta_0^*)}$$

p.f. By Def 133 and Thm 141 □

Thm 144 Pick

$$0 \neq u \in E_0 V_i$$

$$0 \neq v \in E_0^* V$$

Then for $0 \leq i \leq d$,

$$E_i^* u = \frac{\langle u, v \rangle}{\|v\|^2} \sum_{j=0}^d v_j(\theta_j) E_j v, \quad (*)$$

$$E_i v = \frac{\langle u, v \rangle}{\|u\|^2} \sum_{j=0}^d v_j^*(\theta_j^*) E_j^* u \quad (**)$$

pf (*):

$$\begin{aligned}
 E_i^* u &= v_i(A) E_0^* u \\
 &= v_i(A) I E_0^* u \\
 &= v_i(A) \left(\sum_{j=0}^d E_j \right) E_0^* u \\
 &= \sum_{j=0}^d v_i(\theta_j) E_j \underbrace{E_0^* u}_{\| \frac{\langle u, v \rangle}{\|v\|^2} v} \\
 &= \frac{\langle u, v \rangle}{\|v\|^2} \sum_{j=0}^d v_i(\theta_j) E_j v \quad \checkmark
 \end{aligned}$$

(**) sim.

□