

Math 846

Lecture 3

Def 9 A graph $\Gamma = (X, E)$ is called
regular with valency k whenever $|\Gamma(x)| = k \quad \forall x \in X$

Obs if Γ is transitive then Γ is regular.

The graph $\Gamma = H(0, 2)$ is regular with valency $k=0$.

Aside Assume a graph $\Gamma = (X, E)$ is

regular with valency k . Then

(i) Γ has maximal eigenvalue k

(ii) the k -eigenspace has basis

$$\mathbb{1} = \sum_{x \in X} x^{\#}$$

"all 1's vector"

(iii) $\text{mult}(k) = 1$.

pf Γ has eigenvalue k since

$$A\mathbb{1} = k\mathbb{1}.$$

let θ denote any eigenvalue of Γ and pick

$$0 \neq v \in V \text{ st } Av = \theta v.$$

Write

$$v = \sum_{x \in X} \alpha_x x^1 \quad \alpha_x \in \mathbb{F}$$

Define

$$\mu = \max \left\{ |\alpha_x| \mid x \in X \right\}$$

$$\text{obs } \mu \neq 0$$

$$\text{pick } x \in X \text{ st}$$

$$|\alpha_x| = \mu$$

Replacing v by v/α_x if nec, WLOG

$$\alpha_x = 1$$

$$|\alpha_y| \leq 1 \quad \forall y \in X$$

Considering the x -entry of $Av = \theta v$,

$$\sum_{y \in \Gamma(x)} \alpha_y = \theta \alpha_x$$

So

$$\theta = \theta \alpha_x = \sum_{y \in \Gamma(x)} \alpha_y \leq \left| \sum_{y \in \Gamma(x)} \alpha_y \right| \leq \sum_{y \in \Gamma(x)} |\alpha_y| \leq k$$

with equality $\theta = k$ iff $\alpha_y = 1 \quad \forall y \in \Gamma(x)$.

Result follows since Γ is connected. □

DEF 10 A graph $\Gamma = (X, E)$ with diameter D is called distance-regular whenever

for $0 \leq h, i, j \leq D$ and $x, y \in X$ at $d(x, y) = h$,

the number

$$p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is a constant that is independent of x, y

and depends only on h, i, j .

We call the numbers

$$p_{ij}^h \quad (0 \leq h, i, j \leq D)$$

the intersection numbers of Γ .

Assume $\Gamma = (V, E)$ is distance-regular with
diam $D \geq 1$.

Obs
$$p_{ij}^h = p_{ji}^h \quad (0 \leq h, i, j \leq D)$$

By the triangle inequality, the following holds
for $0 \leq h, i, j \leq D$:

- $p_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two.
- $p_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

Write

$$c_i = p_{i, i-1}^i \quad (1 \leq i \leq D), \quad c_0 = 0$$

$$a_i = p_{i, i}^i \quad (0 \leq i \leq D)$$

$$b_i = p_{i, i+1}^i \quad (0 \leq i \leq D-1), \quad b_D = 0$$

We clarify the meaning.

For $x, y \in X$ at $\partial(x, y) = i$,

$$c_i = |\Gamma_{i-1}^-(x) \cap \Gamma(y)|$$

$$a_i = |\Gamma_i^-(x) \cap \Gamma(y)|$$

$$b_i = |\Gamma_{i+1}^-(x) \cap \Gamma(y)|$$

Obs Γ is regular with valency $k = b_0$

Also

$$c_i + a_i + b_i = k \quad (0 \leq i \leq n)$$

Note

$$a_0 = 0, \quad c_1 = 1$$

$$c_i \neq 0 \quad (1 \leq i \leq n),$$

$$b_i \neq 0 \quad (0 \leq i \leq n-1).$$

For $0 \leq i \leq p$ define

$$k_i = p \binom{0}{i}$$

For $x \in X$,

$$k_i = |\Gamma_i(x)|$$

Note

$$k_0 = 1,$$

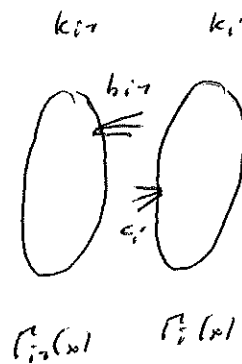
$$k_i = k$$

Note

$$k_i c_i = k_{i-1} b_{i-1}$$

$$(1 \leq i \leq p)$$

Reason:



this yields

$$k_i = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i} \quad (0 \leq i \leq p)$$

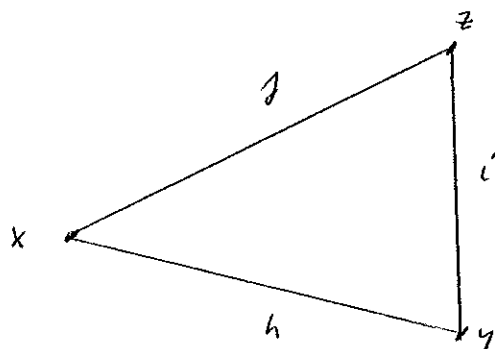
Note $\sum_n 0 \leq h, i, j \leq 0$

$$k_h p_{ij}^h = k_i p_{jh}^i = k_j p_{hi}^j$$

$= |\mathcal{X}|^{-1}$ times the number of 3-tuples

(x, y, z) of vertices such that

$$\partial(x, y) = h, \quad \partial(y, z) = i, \quad \partial(z, x) = j$$

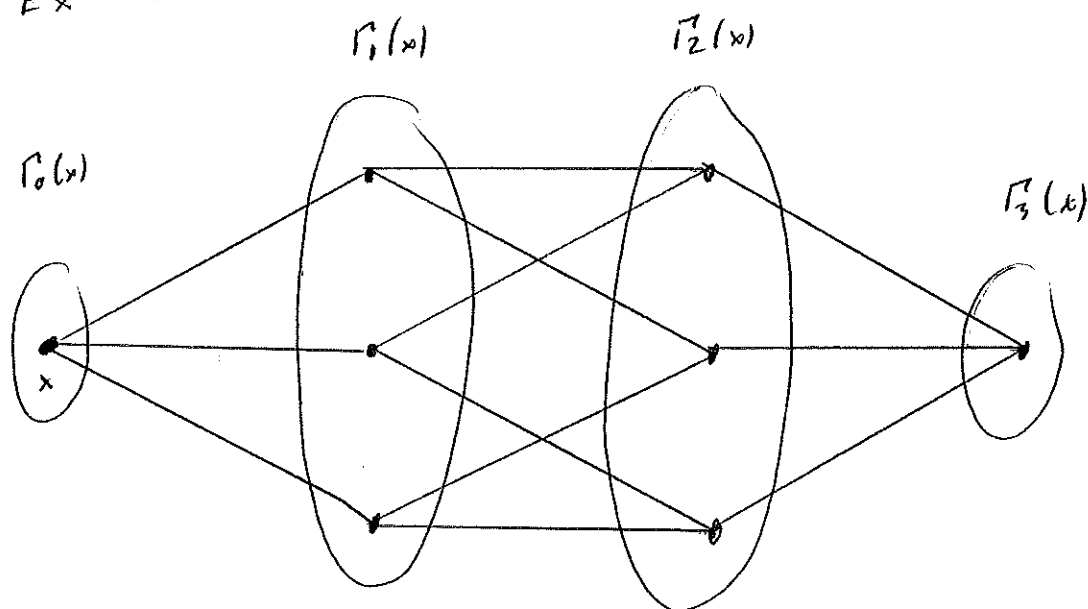


If a graph Γ is distance-transitive
then Γ is distance-regular.

So $\Gamma = H(n, 2)$ is distance-regular.

Next goal: find c_i, a_i, b_i, k_i

Ex $n=3$



$$\begin{aligned} c_0 &= 0 \\ a_0 &= 0 \\ b_0 &= 3 \end{aligned}$$

$$k_0 = 1$$

$$\begin{aligned} c_1 &= 1 \\ a_1 &= 0 \\ b_1 &= 2 \end{aligned}$$

$$k_1 = 3$$

$$\begin{aligned} c_2 &= 2 \\ a_2 &= 0 \\ b_2 &= 1 \end{aligned}$$

$$k_2 = 3$$

$$\begin{aligned} c_3 &= 3 \\ a_3 &= 0 \\ b_3 &= 0 \end{aligned}$$

$$k_3 = 1$$

LEM 11 For $P = H(0, 2)$ and $0 \leq i \leq D$,

$$\begin{aligned} c_i &= i & b_i &= D-i \\ a_i &= 0 & k_i &= \binom{D}{i} \end{aligned}$$

pt Pick $x, y \in X$ at $\partial(x, y) = i$. wlog

$$x = (1, 1, \dots, 1) \quad y = (\underbrace{-1, -1, \dots, -1}_i, 1, 1, \dots, 1)$$

The vertices of $\Gamma_{i-1}(x) \cap \Gamma(y)$ are obtained from y by changing a single -1 to 1 . So

$$|\Gamma_{i-1}(x) \cap \Gamma(y)| = i$$

" c_i

The vertices in $\Gamma_{i+1}(x) \cap \Gamma(y)$ are obtained from y by changing a single 1 to -1 . So

$$|\Gamma_{i+1}(x) \cap \Gamma(y)| = D-i$$

" b_i

Also

$$a_i = k - b_i - c_i = D - (D-i) - i = 0$$

Also

$$k_i = \frac{b_0 b_1 \dots b_{D-1}}{c_0 c_1 \dots c_D} = \binom{D}{i}$$

□

Next goal: describe the algebraic meaning of p_{ij}^h

Until further notice, assume $\Gamma = (X, E)$ is any distance-regular graph (DRG) with diam $D \geq 1$.

Def 12 For $0 \leq i \leq D$ define $A_i \in \text{Mat}_X(\mathbb{F})$ with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad x, y \in X$$

Call A_i the i th distance matrix of Γ .

obs $A_0 = I, \quad A_1 = A, \quad J = \sum_{i=0}^D A_i$

LEM 13 For $0 \leq i, j \leq D$

$$A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$$

p.f. For $x, y \in X$ the (x, y) -entry of each side

is
$$p_{ij}^h \quad h = \partial(x, y)$$

□

LEM 14 We have

- (i) $\{A_i\}_{i=0}^p$ form a basis for M
- (ii) $\dim M = p+1$
- (iii) Γ has exactly $p+1$ eigenvalues

pf (i) the matrices $\{A_i\}_{i=0}^p$ are lin indep
since their non-zero entries are in disjoint locations

Define $m = \text{Span}\{A_i \mid 0 \leq i \leq p\}$

$A_0 = I \in m$ and m is closed under matrix mult
by Lem 13

m is subalgebra of $\text{Mat}_X(\mathbb{F})$

$A_0 = A \in m$ so

$$M \subseteq m$$

$$p+1 \leq \dim M \leq \dim m = p+1$$

$$\text{so } M = m$$

- (ii) By (i)
- (iii) By (ii)

□

Let $\lambda =$ indeterminate

Let $F[\lambda]$ denote the algebra of polynomials in λ that have all coeffs in F

Next goal: for $0 \leq i \leq n$ find $v_i \in F[\lambda]$ s.t.

$$A_i = v_i(A)$$

LEM 15 We have

$$A A_i = b_{i+1} A_{i+1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leq i \leq n-1)$$

$$A A_0 = b_{0+1} A_{0+1} + a_0 A_0$$

p.f. This is Lem 13 with $\gamma = 1$

□

Def 16 We recursively define some

polynomials $\{v_i\}_{i=0}^{b+1}$ in $\mathbb{F}[\lambda]$:

$$v_0 = 1,$$

$$v_1 = \lambda$$

$$\lambda v_i = b_{i+1} v_{i+1} + a_i v_i + c_{i+1} v_{i+1} \quad (1 \leq i \leq b+1)$$

$$\lambda v_b = b_{b+1} v_{b+1} + a_b v_b + v_{b+1}$$

Obs For $0 \leq i \leq b$

v_i has degree i and leading coef $\frac{1}{c_1 c_2 \dots c_i}$

v_{b+1} has degree $b+1$ and leading coef $\frac{1}{c_1 c_2 \dots c_b}$