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Lecture 29

Recall $\mathbb{F}$ arb $d \in \mathbb{N}$ $V = \text{vectn space dim } d+1$ Fix LS on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^x; \{E_i^x\}_{i=0}^d)$$

Define

$$m_i = \text{tr}(E_i E_0^x) \quad 0 \leq i \leq d$$

We saw

$$E_i E_0^x E_i = m_i E_i \quad (0 \leq i \leq d)$$

$$E_0^x E_i E_0^x = m_i E_0^x \quad (0 \leq i \leq d)$$

$$m_i \neq 0 \quad (0 \leq i \leq d)$$

$$\sum_{i=0}^d m_i = 1$$

$$m_0 = m_0^x$$

Def 117 For $m_0 = m_0^*$ let ν

denote the inverse of this common value. So

$$\nu = \nu^*$$

By const.

$$\text{tr}(E_0 E_0^*) = \nu^{-1}$$

We call ν the size of Φ

LEM 118 We have

$$(i) \quad \nu E_0 E_0^* E_0 = E_0$$

$$(ii) \quad \nu E_0^* E_0 E_0^* = E_0^*$$

pf (i) By Lem 116

$$E_0 E_0^* E_0 = m_0 E_0$$

" ν^{-1}

(ii) Sim.

□

Pick $0 \neq u \in E_0 V$

Recall the Φ -standard basis for V :

$$E_i^* u \quad 0 \leq i \leq d$$

We give some characterizations of this basis

Ex 119 Let $\{v_i\}_{i=0}^d$ denote any vectors in V ,
not all 0. Show that $\{v_i\}_{i=0}^d$ is a Φ -standard
basis for V iff both

$$(i) \quad v_i \in E_i^* V \quad (0 \leq i \leq d)$$

$$(ii) \quad \sum_{i=0}^d v_i \in E_0 V$$

Ex 120 Let $\{v_i\}_{i=0}^d$ denote any basis for V .

Let B (resp. B^*) denote the matrices in $\text{Mat}_d(\mathbb{F})$

that represent A (resp. A^*) wrt $\{v_i\}_{i=0}^d$.

Then $\{v_i\}_{i=0}^d$ is a $\mathbb{F}$ -standard basis for V

iff both

(i) B has constant row sum θ_0

(ii) $B^* = \text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$

DEF 121 For $X \in \text{End}(V)$ let X^b denote the matrix in $\text{Mat}_d(\mathbb{F})$ that represents X wrt a $\mathbb{F}$ -standard basis for V . Note that

$$b: \begin{array}{ccc} \text{End}(V) & \rightarrow & \text{Mat}_d(\mathbb{F}) \\ X & \rightarrow & X^b \end{array}$$

is an algebra isomorphism.

We have

$$A^b = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & & \ddots & \\ 0 & & & & b_{d-1} \\ & & & c_d & a_d \end{pmatrix}$$

$$(A^*)^b = \text{diag}(a_0^*, a_1^*, \dots, a_d^*)$$

$$(E_i^*)^b = \text{diag}(\underbrace{0 \dots 0}_{i \text{ times}}, 1, \underbrace{0 \dots 0}_{d-i \text{ times}})$$

as is

Note that

$$b_i, c_i = x_i$$

(is is)

LEM 122

 $F_n \quad 0 \leq i \leq dH,$

$$b_0 b_1 \dots b_{i-1} = p_i(a_0)$$

pf Assume $0 \leq i \leq d$, else each side is 0.

Obs $b_0 b_1 \dots b_{i-1} = (0, i) \text{-entry of } (A^i)^b$

Pick $0 \neq v \in E_0 V$. So

$$E_0^* v, E_1^* v, \dots, E_d^* v$$

is $\mathbb{F}$ -stand basis for V .

By const

$$E_0^* A^i E_i^* v = b_0 b_1 \dots b_{i-1} E_0^* v$$

By Prop 112

$$E_i^* A^i E_0^* = p_i(A) E_0^*$$

Apply $\dagger$:

$$E_0^* A^i E_i^* = E_0^* p_i(A)$$

So

$$\begin{aligned} b_0 b_1 \dots b_{i-1} E_0^* v &= E_0^* A^i E_i^* v = E_0^* p_i(A) v \\ &= p_i(a_0) E_0^* v \end{aligned}$$

Result follows. □

(1) Prop 123 We have

$$p_i(\theta_0) \neq 0 \quad (0 \leq i \leq d)$$

Moreover

$$b_i = \frac{p_{i-1}(\theta_0)}{p_i(\theta_0)} \quad 0 \leq i \leq d$$

$$c_i = \frac{X_i p_{i-1}(\theta_0)}{p_i(\theta_0)} \quad (1 \leq i \leq d)$$

(2) pf By Lem 122 and

$$b_{i-1} c_i = X_i \quad (1 \leq i \leq d)$$

□

Prop 124 We have

$$(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d) = \gamma c_1 c_2 \cdots c_d$$

p.f. Recall

$$E_0 = \prod_{j=1}^d \frac{A - \theta_j I}{\theta_0 - \theta_j}$$

So

$$(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d) E_0 = (A - \theta_1 I) \cdots (A - \theta_d I)$$

consider

$$E_d^* (\cdots) E_0^*$$

$$\begin{aligned} (\theta_0 - \theta_1) \cdots (\theta_0 - \theta_d) E_d^* E_0 E_0^* &= E_d^* \left((A - \theta_1 I) \cdots (A - \theta_d I) \right) E_0^* \\ &= E_d^* A^d E_0^* \end{aligned}$$

Put $0 \neq v \in E_0 V$

Use $\mathbb{F}$ -standard basis

$$E_0^* v, E_1^* v, \dots, E_d^* v$$

$$c_1, c_2, \dots, c_d = (d, d)\text{-entry of } (A^d)^b$$

So

$$c_1, c_2, \dots, c_d \ E_d^* V = E_d^* A^d E_0^* V$$

We have

$$\begin{aligned} (c_0 - a_1) \dots (c_0 - a_d) \ E_d^* E_0 E_0^* V &= E_d^* A^d E_0^* V \\ &= c_1, c_2, \dots, c_d \ E_d^* V \end{aligned}$$

$$\begin{aligned} E_0 E_0^* V &= E_0 E_0^* E_0 V \\ &= V^{-1} E_0 V \\ &= V^{-1} V \end{aligned}$$

So

$$(c_0 - a_1) \dots (c_0 - a_d) V^{-1} E_d^* V = c_1, c_2, \dots, c_d \ E_d^* V$$

Result follows

□

Def 125 Define

$$k_i = m_i^* v \quad (0 \leq i \leq d)$$

LEM 126 We have

$$(i) \quad k_0 = 1$$

$$(ii) \quad k_i \neq 0 \quad (0 \leq i \leq d)$$

$$(iii) \quad v = \sum_{i=0}^d k_i$$

$$\text{pf (i)} \quad k_0 = \underset{\substack{= \\ v^{-1}}}{m_0^*} v = 1$$

$$(ii) \quad m_i^* \neq 0 \text{ and } v \neq 0$$

(iii) Since

$$\sum_{i=0}^d m_i = 1$$

□

LEM 127 For $0 \leq i \leq d$,

$$k_i = \frac{p_i(\theta_0)^2}{x_1 x_2 \cdots x_i}$$

pf We have

$$p_i(A|E_0^x) = E_i^x A^i E_0^x$$

Apply $\dagger$:

$$E_0^x p_i(A) = E_0^x A^i E_i^x$$

$$\begin{aligned} \text{So } p_i(A) E_0^x p_i(A) &= E_i^x A^i E_0^x A^i E_i^x \\ &= x_1 x_2 \cdots x_i E_i^x \end{aligned}$$

$$\text{So } p_i(A) E_0^x p_i(A) E_0 = x_1 x_2 \cdots x_i E_i^x E_0$$

Take trace:

$$\left(p_i(\theta_0) \right)^2 \underbrace{\text{tr}(E_0^x E_0)}_{\substack{n \\ \vee^{-1}}} = x_1 x_2 \cdots x_i m_i^x$$

$$\begin{aligned} \text{So } \left(p_i(\theta_0) \right)^2 &= x_1 x_2 \cdots x_i \vee m_i^x \\ &= x_1 x_2 \cdots x_i k_i \end{aligned}$$

□

Prop 128

For $0 \leq i \leq d$,

$$k_i = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$$

p f obs

$$k_i = \frac{(p_i(a_0))^2}{x_1 x_2 \dots x_i}$$

$$= \frac{(b_0 b_1 \dots b_{i-1})^2}{(b_0 c_1)(b_1 c_2) \dots (b_{i-1} c_i)}$$

$$= \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$$

□

Recall the polynomials $\{p_i\}_{i=0}^d$ We now
adjust the normalization

DEF 129 define the polynomial

$$v_i = \frac{p_i}{c_1 c_2 \dots c_i} \quad (0 \leq i \leq d)$$

obs $v_0 = 1$

LEM 130 $\forall n$ $0 \leq i \leq d$,

$$v_i(0) = k_i$$

pf $v_i(0) = \frac{p_i(0)}{c_1 c_2 \dots c_i} = \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i} = k_i$ □

LEM 131

 $\forall n \quad 0 \leq i \leq d-1$

$$\lambda v_i = c_{in} v_{in} + a_i v_i + b_{i+1} v_{i+1}$$

where $b_{-1} = 0$ and $v_{-1} = 0$

Moreover

$$\lambda v_d - a_d v_d - b_{d+1} v_{d+1} = \frac{p_{dn}}{c_1 c_2 \dots c_d}$$

pf $\forall n \quad 0 \leq i \leq d$

$$\lambda p_i = p_{in} + a_i p_i + x_i p_{i+1}$$

divide each side by $c_1 c_2 \dots c_i$ and use Def 129. $\square$

Pick $u \in E_0 V$ and recall the $\mathbb{F}$ -standard basis for V :

$$E_0^* u, E_1^* u, \dots, E_n^* u.$$

Thm 132 With above notation.

$$V_i(A) E_0^* u = E_i^* u \quad (\text{desired}).$$

pf Recall

$$P_i(A) E_0^* V = E_i^* V$$

so $\exists \alpha \in \mathbb{F}$ st

$$V_i(A) E_0^* u = \alpha E_i^* u$$

show $\alpha = 1$.

obs

$$V_i(A) E_0^* E_0 = \alpha E_i^* E_0$$

Take trace:

$$\underbrace{V_i(A)}_{\text{"}} \underbrace{V^{-1}}_{k_i} = \alpha \underbrace{m_i^*}_{\text{"}}.$$

$$\text{so } \alpha = 1 \quad \checkmark$$

□

Here is another normalization of the polynomials $\{p_i\}_{i=0}^d$

DEF 133 Define the polynomial

$$u_i = \frac{p_i}{p_i(0)} \quad (0 \leq i \leq d)$$

We obs $u_0 = 1$ and

$$u_i(0) = 1 \quad (0 \leq i \leq d)$$

LEM 134

For $0 \leq i \leq d-1$,

$$\lambda u_i = b_i u_{i+1} + a_i u_i + c_i u_{i-1}$$

where $u_{-1} = 0$.

Moreover

$$\lambda u_d - a_d u_d - c_d u_{d-1} = \frac{p_d \pi}{p_d(\theta_0)}$$

pf For $0 \leq i \leq d$

$$\lambda p_i = p_{i+1} + a_i p_i + c_i p_{i-1}$$

Divide both sides by $p_i(\theta_0)$ and use def 133.

□