

Math 846

Lecture 28

Next goal: explain what the Leonard systems have to do with the classical orthogonal polynomials.

Recall: $\mathbb{F}$ a field $d \in \mathbb{N}$

$V =$ vector space $\dim d+1$

Fix LS on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Notation For any object f attached to Φ ,

f^* denotes the conjugate object for Φ^*

$f^\downarrow \quad \dots$

$\Phi^\downarrow$

$f^\uparrow \quad \dots$

$\Phi^\uparrow$

Recall $\text{End}(V)$ has a basis

$$A^r E_0^* A^s \quad 0 \leq r, s \leq d$$

Recall

$$\mathcal{D} = \text{subalg of } \text{End}(V) \text{ gen by } A$$

$$\{A^i\}_{i=0}^d \text{ is a basis for } \mathcal{D}$$

Let $\{X_i\}_{i=0}^d$ denote any basis for $\mathcal{D}$.

Then $\text{End}(V)$ has a basis

$$X_i E_0^* X_j \quad (0 \leq i, j \leq d)$$

Recall $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{D}$.

So $\text{End}(V)$ has a basis

$$E_i E_0^* E_j \quad (0 \leq i, j \leq d)$$

(1) LEM 104 For $0 \leq i, j \leq d$ the following hold

$$(i) \quad E_i^* A^r E_j^* = 0 \text{ if } r < |i-j| \quad (0 \leq r \leq d)$$

(ii) Suppose $i \leq j$. Then

$$E_i^* A^{j-i} E_j^* = E_i^* A E_{i+1}^* A \cdots A E_j^* \neq 0$$

(iii) Suppose $i \geq j$. Then

$$E_i^* A^{i-j} E_j^* = E_i^* A E_{i-1}^* A \cdots A E_j^* \neq 0$$

pf Represent A (resp A^*) by an
upper triangular (resp. diagonal) matrix in
 $\text{Mat}_d(\mathbb{F})$.

□

Recall

$$a_i = \text{tr} (A E_i^k) \quad (0 \leq i \leq d)$$

Define

$$x_i = \text{tr} (A E_{i-1}^k A E_i^k) \quad (1 \leq i \leq d)$$

and

$$x_0 = 0.$$

For $0 \leq i \leq d$ pick $0 \neq v_i \in E_i^k V$

So $\{v_i\}_{i=0}^d$ is basis for V

With respect to this basis E_i^k is represented by

$$\text{diag} (0 \cdots 0 \underset{\substack{\uparrow \\ i}}{1} 0 \cdots 0) \quad (0 \leq i \leq d)$$

Let $B \in \text{Mat}_d(\mathbb{F})$ represent A wrt $\{v_i\}_{i=0}^d$.

Then B is irred tridiagonal. Moreover

$$B_{ii} = a_i \quad (0 \leq i \leq d)$$

$$B_{i,i-1} B_{i-1,i} = x_i \quad (1 \leq i \leq d)$$

$$x_i \neq 0 \quad (1 \leq i \leq d)$$

LEM 105 Pick $0 \neq v \in E_0^* V$,

then for $0 \leq i \leq d$ the vector

$$E_i^* A^i v$$

is non 0 and hence a basis for $E_i^* V$

Moreover

$$\{ E_i^* A^i v \}_{i=0}^d$$

is a basis for V

pf For $0 \leq i \leq d$ the product

$$E_i^* A^i E_0^* \neq 0$$

by LEM 104. Rest is clear. □

LEM 106 With resp to the basis for V
 from LEM 105, A is represented by the
 following matrix in $\text{Mat}_d(\mathbb{F})$:

$$\begin{pmatrix} a_0 x_1 & & & 0 \\ 1 & a_1 x_2 & & \\ & 1 & a_2 & \ddots \\ 0 & & \ddots & x_d \\ & & & 1 & a_d \end{pmatrix}$$

pf Call the basis $\{v_i\}_{i=0}^d$ and the representing
 matrix B . So

$$Av_j = \sum_{i=0}^d B_{ij} v_i \quad (0 \leq j \leq d)$$

B is irred tri diag. Show

$$B_{i,i-1} = 1 \quad (1 \leq i \leq d)$$

We have

$$v_i = E_i^* A^i E_0^* \quad (0 \leq i \leq d)$$

In matrix terms, this becomes

$$i \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \text{diag}(0 \dots 0 \underset{i}{1} 0 \dots 0) B^i \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{So } 1 = B_{i,i-1} \dots B_{2,1} B_{1,0} \quad (1 \leq i \leq d)$$

So

$$B_{i,i-1} = 1$$

$(1 \leq i \leq d)$ Rest is clear $\square$

LEM 107 For $1 \leq i \leq d$,

$$E_i^* A E_i^* A E_i^* = x_i E_i^*$$

$$E_{i-1}^* A E_i^* A E_i^* = x_i E_{i-1}^*$$

pf, Represent A and $\{E_j^*\}_{j=0}^d$ by matrices wrt the basis in LEM 105. □

LEM 108 For $0 \leq i \leq j \leq d$,

$$E_j^* A^{j-i} E_i^* A^{j-i} E_j^* = x_{i+1} \cdots x_{j-1} x_j E_j^*,$$

$$E_i^* A^{j-i} E_j^* A^{j-i} E_i^* = x_{i+1} \cdots x_{j-1} x_j E_i^*$$

pf By Lem 104, 107 and induction on $j-i$. □

DEF 109 We define polynomials

$$p_i \in F[\lambda] \quad (0 \leq i \leq d+n)$$

by

$$p_0 = 1,$$

$$\lambda p_i = p_{i+n} + a_i p_i + x_i p_{i-1} \quad (0 \leq i \leq d)$$

where $p_{-1} = 0$

Obs

p_i is monic with degree i $(0 \leq i \leq d+n)$.

LEM 110 Pick $0 \neq v \in E_0^* V$. Then

$$p_i(A)v = E_i^* A' v \quad (0 \leq i \leq d),$$

$$p_{d+1}(A)v = 0$$

pf Write

$$v_i = p_i(A)v \quad (0 \leq i \leq d+1)$$

By Def 109.

$$v_0 = v$$

$$Av_i = v_{i+1} + a_i v_i + x_i v_{i-1} \quad (0 \leq i \leq d)$$

where $v_{-1} = 0$.

Write

$$v'_i = E_i^* A' v \quad (0 \leq i \leq d)$$

$$v'_{d+1} = 0$$

By LEM 106

$$v_0' = v$$

$$Av'_i = v'_{i+1} + a_i v'_i + x_i v'_{i-1} \quad (0 \leq i \leq d)$$

where $v'_{-1} = 0$.

It follows

$$v_i = v'_i \quad (0 \leq i \leq d+1)$$

□

Prop 111 $\forall n \ 0 \leq i \leq d,$

$$p_i(A) E_0^* V = E_i^* V$$

pf We refer to Lem 110

$$p_i(A) v = E_i^* A^i v$$

v is basis for $E_0^* V$

$E_i^* A^i v$ is basis for $E_i^* V$

Result follows.

□

Prop 112 F_n oried

$$p_i(A) E_0^* = E_i^* A^i E_0^*$$

pt we refer to Lem 110

$$p_i(A)v = E_i^* A^i v$$

$$\text{So } (p_i(A) - E_i^* A^i) v = 0$$

v is basis for $E_0^* V$

$$\text{So } (p_i(A) - E_i^* A^i) E_0^* = 0$$

□

Prop 113 We have

(i) p_{dn} is both the minimal and characteristic poly of A

$$(ii) \quad p_{\text{dn}} = (\lambda - \alpha_0)(\lambda - \alpha_1) \cdots (\lambda - \alpha_d)$$

pf (i) Recall $\{A^i\}_{i=0}^d$ is a basis for the subalg of $\text{End}(V)$ gen by A .

Remains to show

$$p_{\text{dn}}(A) = 0$$

By Lem 110,

$$p_{\text{dn}}(A) E_0^* V = 0$$

We need

$$p_{\text{dn}}(A) V = 0$$

Write
$$V = \sum_{i=0}^d E_i^* V$$

For $0 \leq i \leq d$,

$$\begin{aligned} p_{\text{dn}}(A) E_i^* V &= p_{\text{dn}}(A) p_i(A) E_0^* V \\ &= p_i(A) \underbrace{p_{\text{dn}}(A)}_{=0} E_0^* V \\ &= 0 \end{aligned}$$

So

$$p_{\text{dn}}(A) V = 0$$

(ii) $\{\alpha_i\}_{i=0}^d$ are the eigenvalues of A

□

() LEM 114 For orisied

$$E_i^* = \frac{P_i(A) E_0^* P_i(A)}{x_1 x_2 \dots x_i}$$

pt By Lem 112,

$$P_i(A) E_0^* = E_i^* A' E_0^*$$

Apply adjoint ^{t} :

$$E_0^* P_i(A) = E_0^* A' E_i^*$$

Now using Lem 108,

$$\begin{aligned} P_i(A) E_0^* P_i(A) &= E_i^* A' E_0^* A' E_i^* \\ &= x_1 x_2 \dots x_i E_i^* \end{aligned}$$

Result follows. □

DEF 115 define

$$m_i = \text{tr}(E_i E_0^*) \quad (0 \leq i \leq d)$$

(1) LEM 116 We have

$$(i) \quad E_i E_0^* E_i = m_i E_i \quad (0 \leq i \leq d)$$

$$(ii) \quad E_0^* E_i E_0^* = m_i E_0^* \quad (0 \leq i \leq d)$$

$$(iii) \quad m_i \neq 0$$

$$(iv) \quad \sum_{i=0}^d m_i = 1$$

$$(v) \quad m_0 = m_0^*$$

pf (i) Abbr $A = \text{End}(V)$
 $E_i A E_i = \text{Span}(E_i)$

$\exists \alpha_i \in \mathbb{F}$ st

$$E_i E_0^* E_i = \alpha_i E_i$$

take trace to get $\alpha_i = m_i$

(ii) Similar to (i)

(iii) Recall $\{E_i E_0^* E_i\}_{0 \leq i \leq d}$ is basis for $\text{End}(V)$

$$\text{So } E_i E_0^* E_i \neq 0$$

$$(iv) \quad \sum_{i=0}^d m_i = \text{tr} \left(\sum_{i=0}^d E_i E_0^* \right) = \text{tr}(I E_0^*) = \text{tr}(E_0^*) = 1$$

(v) $E_0 E_0^*$ and $E_0^* E_0$ have same trace

□