

Math 846

Lecture 27

We finished proving the classification Thm 82
for LS

Next goal: give some characterizations of
Leonard systems and parameter arrays.

Recall: $\mathbb{F}$ arb, $d \in \mathbb{N}$

$$V = V^S \text{ dim } d+1$$

Notation: Given two bases for V ,

$$\{u_i\}_{i=0}^d$$

$$\{v_i\}_{i=0}^d$$

By the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$

we mean the matrix $B \in \text{Mat}_{d+1}(\mathbb{F})$ s.t.

$$v_j = \sum_{i=0}^d B_{ij} u_i \quad 0 \leq j \leq d$$

Thm 93 let

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

denote a pre Leonard system on V , with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$

then Φ is a Leonard system on V iff the following (i), (ii) hold:

(i) $\exists$ nmo scalars $\{\varphi_i\}_{i=1}^d$ in $\mathbb{F}$ and a basis for V wrt which

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & 1 & \ddots & \\ & & \ddots & \theta_d \\ 0 & & & 1 & \theta_d \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & \varphi_2 & \dots & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ 0 & & & \varphi_d & \\ & & & & \theta_d^* \end{pmatrix}$$

(ii) $\exists$ nmo scalars $\{\phi_i\}_{i=1}^d$ in $\mathbb{F}$ and a basis for V wrt which

$$A: \begin{pmatrix} \theta_d & & & 0 \\ & \theta_{d-1} & & \\ & 1 & \ddots & \\ & & \ddots & \theta_0 \\ 0 & & & 1 & \theta_0 \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & \dots & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ 0 & & & \phi_d & \\ & & & & \theta_d^* \end{pmatrix}$$

In this case Φ has parameter array
 $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$

pf Assume (i), (ii). We show Φ is LS on V

We invoke the 3/4 thm. Prop 5.6.

consider the conditions

(1) for $0 \leq i, j \leq d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

(2) for $0 \leq i, j \leq d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases}$$

(3) for $0 \leq i, j \leq d$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

(4) for $0 \leq i, j \leq d$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases}$$

Apply Prop 69 and Lem 70 to Φ , gives

(1), (4)

Apply Prop 69 and Lem 70 to $\Phi^{\vee}$, gives

(1), (3)

We have (1), (3), (4)

By 3/4 thm (2) holds.

So Φ is LS in V

The thm is proved in one direction.
The other direction is clear.

The last assertion is clear.

□

Thm 94 Consider a sequence of scalars in $\mathbb{F}$

$$\left(\{\theta_i\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^d ; \{\psi_i\}_{i=1}^d ; \{\phi_i\}_{i=1}^d \right) \quad (\star)$$

that satisfies (PA1), (PA2).

FAE

(i) $(\star)$ satisfies (PA3) - (PA5)

(ii) $\exists$ invertible $G \in \text{Mat}_{d+1}(\mathbb{F})$ s.t. both

$$G^{-1} \begin{pmatrix} \theta_0 & \theta_1 & & & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & \theta_d \end{pmatrix} G = \begin{pmatrix} \theta_d & & & & 0 \\ & \theta_{d-1} & & & \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & \theta_0 \end{pmatrix},$$

$$G^{-1} \begin{pmatrix} \theta_0^* & \psi_1 & & & 0 \\ \theta_1^* & \psi_2 & & & \\ & & \ddots & & \\ 0 & & & \psi_d & \\ & & & \theta_d^* & \end{pmatrix} G = \begin{pmatrix} \theta_0^* & \phi_1 & & & 0 \\ \theta_1^* & \phi_2 & & & \\ & & \ddots & & \\ 0 & & & \phi_d & \\ & & & \theta_d^* & \end{pmatrix}.$$

pf (i) $\rightarrow$ (ii) $(\star)$ is a param array

so $\exists$ LS Φ on V with param array $(\star)$

By Prop 69 $\exists$ basis $\{u_i\}_{i=0}^d$ of V

wrt which:

$$A: \begin{pmatrix} \theta_0 & \theta_1 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \theta_d \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \theta_{d-1}^* & \\ & 0 & & & \theta_d^* \end{pmatrix}$$

By Lem 75 $\exists$ basis $\{v_i\}_{i=0}^d$ of V

wrt which:

$$A: \begin{pmatrix} \theta_d & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \theta_0 \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \theta_{d-1}^* & \\ & 0 & & & \theta_d^* \end{pmatrix}$$

let G denote transim matrix for $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$
 then (ii) holds by linear algebra.

(ii) $\rightarrow$ (i) Put a basis $\{u_i\}_{i=0}^d$ for V

Define $A, A^* \in \text{End}(V)$ that are represented as follows wrt $\{u_i\}_{i=0}^d$:

$$A: \begin{pmatrix} \theta_0 & & & & 0 \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & & & & 0 \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \theta_d^* \end{pmatrix}$$

Obs

- A is mult free with eigenvalues $\{\theta_i\}_{i=0}^d$
- A^* is mult free with eigenvalues $\{\theta_i^*\}_{i=0}^d$
- $\forall \alpha \ 0 \leq i \leq d$ let E_i (resp E_i^*) denote the prim idempotent of A (resp A^*) for θ_i (resp θ_i^*)

the sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a LS in V

Since G is invertible, $\exists$ basis $\{v_i\}_{i=0}^d$ for V st G is transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$

By lin algebra, A and A^* look as follows
wrt $\{v_i\}_{i=0}^d$:

$$A = \begin{pmatrix} \phi_0 & & & 0 \\ 1 & \phi_1 & & \\ & 1 & \ddots & \\ 0 & & \ddots & 1 & \phi_d \end{pmatrix}$$

$$A^* = \begin{pmatrix} \phi_0^* & \phi_1 & \phi_2 & 0 \\ & \phi_1^* & \ddots & \\ 0 & & \ddots & \phi_d \\ & & & \phi_d^* \end{pmatrix}$$

Now by Thm 93,

Φ is LS on V with param array $(\star)$.

In particular $(\star)$ is a param array so it satisfies
(PA3) - (PA5)

□

Given LS on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

In the previous results, we represented A and A^* by a lower bidiagonal matrix and upper bidiagonal matrix.

Next, we represent one of A, A^* by an unred. tridiagonal matrix and the other by a diagonal matrix.

By Lem 68, E_0^* is normalizing.

Similarly E_0 is normalizing

Since E_0 is normalizing, for $0 \neq z \in E_0 V$ the vectors

$$\{E_i^* z\}_{i=0}^d$$

form a basis for V .

Call this basis Φ -standard

With respect to a $\mathbb{F}$ -standard basis for V ,
the matrices rep A and A^v are

$$A: \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & & \ddots & \\ & & 0 & & b_{d-1} \\ & & & c_d & a_d \end{pmatrix}$$

$$A^v: \text{diag}(\theta_0^v, \theta_1^v, \dots, \theta_d^v)$$

where $\{a_i\}_{i=0}^d$ are from Def 50 and

$$0 \neq b_i \in \mathbb{F} \quad (0 \leq i \leq d-1)$$

$$0 \neq c_i \in \mathbb{F} \quad (1 \leq i \leq d)$$

LEM 95 With above notation,

$$c_i + a_i + b_i = \theta_0 \quad (0 \leq i \leq d)$$

where $c_0 = 0, \quad b_d = 0$

pf $Az = \theta_0 z \quad z = Iz = \sum_{i=0}^d E_i^v z$

$$\text{so } \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & & \ddots & \\ & & 0 & & b_{d-1} \\ & & & c_d & a_d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \theta_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Result follows. □

We now dualize.

Since E_0^v is normalizing, $\nexists 0 \neq z^* \in E_0^{*v}$

The vectors

$$\{E_i z^*\}_{i=0}^d$$

form a basis for V

call this basis Φ^x -standard

With respect to a Φ^x -standard basis for V ,

$$A: \text{diag}(a_0, a_1, \dots, a_d), \quad A^x: \begin{pmatrix} a_0^x & b_0^v & & & \\ c_1^x & a_1^x & b_1^x & & \\ & c_2^x & & \ddots & \\ 0 & & & & b_{d-1}^v \\ & & & c_d^x & a_d^x \end{pmatrix}$$

where $\{a_i^x\}_{i=0}^d$ are from def 50 and

$$0 \neq b_i^v \in \mathbb{F}$$

$$(0 \leq i \leq d-1)$$

$$0 \neq c_i^x \in \mathbb{F}$$

$$(1 \leq i \leq d)$$

Applying Lem 95 to Φ^x ,

$$c_i^x + a_i^x + b_i^x = \theta_0^x \quad (0 \leq i \leq d)$$

$$\text{where } c_0^x = 0, \quad b_d^v = 0$$

We call

$$\{c_i\}_{i=1}^d, \{a_i\}_{i=0}^d, \{b_i\}_{i=0}^{d-1}$$

the intersection numbers of Φ

We call

$$\{c_i^x\}_{i=1}^d, \{a_i^x\}_{i=0}^d, \{b_i^x\}_{i=0}^{d-1}$$

the dual intersection numbers of Φ .

LEM 96 Assume $d \geq 1$. For the LS $\mathbb{F}$

and $0 \leq i \leq d$,

$$c_i \theta_{i+1}^* + a_i \theta_i^* + b_i \theta_{i-1}^* = \theta_i \theta_i^* + a_0^* (\theta_0 - \theta_i),$$

$$c_i^* \theta_{i+1} + a_i^* \theta_i + b_i^* \theta_{i-1} = \theta_i^* \theta_i + a_0 (\theta_0^* - \theta_i^*)$$

where $\theta_{-1}, \theta_{d+1}, \theta_1^*, \theta_{d+1}^*$ are indeterminates.

pf We prove lit eqn. Recall

$$A^* E_0 V \leq E_0 V + E_1 V,$$

$$E_0 A^* E_0 = a_0^* E_0$$

$$\text{So } (A^* - a_0^* I) E_0 V \leq E_1 V$$

$$\text{So } (A - \theta_1 I)(A^* - a_0^* I) E_0 V = 0$$

In matrix terms,

$$\begin{pmatrix} a_0 - \theta_1 & b_0 & & & \\ c_1 & a_1 - \theta_1 & b_1 & & \\ & c_2 & & \ddots & \\ & & & \ddots & b_{d-1} \\ & & & & c_d & a_d - \theta_1 \end{pmatrix} \begin{pmatrix} \theta_0^* - a_0^* & & & & \\ & \theta_1^* - a_0^* & & & \\ & & \theta_2^* - a_0^* & & \\ & & & \ddots & \\ & & & & \theta_d^* - a_0^* \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

Result follows

□

LEM 97 Assume $d \geq 1$. For the

LS $\mathbb{F}$,

$$b_0 = \theta_0 - a_0$$

$$b_i = \frac{(a_i - \theta_0)(\theta_i^* - \theta_{i+1}^*) + (\theta_0 - \theta_i)(\theta_i^* - a_0^*)}{\theta_{i+1}^* - \theta_{i+1}^*}$$

$$(1 \leq i \leq d-1)$$

$$c_i = \frac{(a_i - \theta_0)(\theta_i^* - \theta_{i+1}^*) + (\theta_0 - \theta_i)(\theta_i^* - a_0^*)}{\theta_{i+1}^* - \theta_{i+1}^*}$$

$$(1 \leq i \leq d-1)$$

$$c_d = \theta_0 - a_d$$

To get $\{b_i^*\}_{i=0}^{d-1}$ and $\{c_i^*\}_{i=1}^d$, interchange
 $\text{star} \leftrightarrow \text{no star}$ everywhere.

pf For $1 \leq i \leq d-1$, solve for b_i, c_i using

Lem 95 and Lem 96. The remaining
 assertions are clear. $\square$

For the LS $\mathbb{F}$

In Lem 71 or Lem 77 we expressed

$\{a_i\}_{i=0}^d$, $\{a_i^*\}_{i=0}^d$ in terms of the param

array of $\mathbb{F}$.

In Lem 97 we express

$\{b_i\}_{i=0}^{d-1}$, $\{c_i\}_{i=1}^d$, $\{b_i^*\}_{i=0}^{d-1}$, $\{c_i^*\}_{i=1}^d$

in terms of the param array of $\mathbb{F}$

In the Appendix to the paper

Notes on the LS classification

all the parameter arrays are given in closed form

For each param array, the corresp intersection numbers and dual intersection numbers are given

The param arrays come in 13 infinite families

Next general goal: Explain what these

13 families have to do with the classical orthogonal polynomials.

Ex 98 Given $d \in \mathbb{N}$ and $A, A^* \in \text{Mat}_d(\mathbb{F})$

Assume A is lower bidiagonal and A^* is upper bidiagonal.

Show TFAE:

(i) A, A^* is Leonard pair

(ii) $\exists$ param array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

over $\mathbb{F}$ s.t

$$A_{ii} = \theta_i, \quad A_{ii}^* = \theta_i^* \quad (0 \leq i \leq d)$$

$$A_{i,i+1} A_{i+1,i}^* = \varphi_i \quad (1 \leq i \leq d)$$

$$\lambda = \text{indet}$$

Ex 99 Given $d \in \mathbb{N}$ and scalars

$$\left(\{\theta_i\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^d ; \{\psi_i\}_{i=1}^d ; \{\phi_i\}_{i=1}^d \right) \quad (*)$$

in F that satisfy PA1, PA2.

Show TFAE:

(i) the sequence $(*)$ satisfies PA3-PA5

(ii) for $0 \leq i \leq d$ the polynomial

$$\sum_{h=0}^i \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{h-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{h-1}^*)}{\psi_1 \psi_2 \cdots \psi_h}$$

is a scalar multiple of the polynomial

$$\sum_{h=0}^i \frac{(\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-h+1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{h-1}^*)}{\phi_1 \phi_2 \cdots \phi_h}$$

We showed that any TD pair A, A^* satisfies the tridiagonal relations TD1, TD2.

If A, A^* is a Leonard pair then we can improve this result as follows.

EX 100 Given $d \in \mathbb{N}$ and V s.t. $\dim V = d$.

Given LP A, A^* on V . Show $\exists$ scalars

$\beta, \gamma, \gamma^*, \delta, \delta^*, \omega, \eta, \eta^* \in \mathbb{F}$ s.t.

$$A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* = \gamma^* A^2 + \omega A + \eta I,$$

$$A^{*2} A - \beta^* A^* A A^* + A A^{*2} - \gamma^* (A^* A + A A^*) - \delta^* A = \gamma A^{*2} + \omega A^* + \eta^* I.$$

These scalars are unique if $d \geq 3$.

The above equations are called the

Askey - Wilson Relations

Ex 10 Assume $\text{char}(\mathbb{F}) = 0$ Fix $d \in \mathbb{N}$

Define $A, A^* \in \text{Mat}_d(\mathbb{F})$ by

$$A = \begin{pmatrix} 0 & d & & 0 \\ 1 & 0 & d-2 & \\ & 2 & 0 & \ddots \\ 0 & \ddots & \ddots & 1 \\ & & d & 0 \end{pmatrix}$$

$$A^* = \text{diag}(d, d-2, d-4, \dots, -d)$$

(i) Find a nonsingular $S \in \text{Mat}_d(\mathbb{F})$ and scalars $\{\psi_i\}_{i=1}^d$ in $\mathbb{F}$ st

$$S^{-1} A S = \begin{pmatrix} d & & 0 \\ 1 & d-2 & \\ & 1 & d-4 \\ 0 & \ddots & \ddots \\ & & 1 & -d \end{pmatrix}$$

$$S^{-1} A^* S = \begin{pmatrix} d & \psi_1 & & 0 \\ & d-2 & \psi_2 & \\ & & d-4 & \ddots \\ 0 & \ddots & \ddots & \psi_d \\ & & & -d \end{pmatrix}$$

(ii) Find a nonsingular $T \in \text{Mat}_d(\mathbb{F})$
and scalars $\{\phi_i\}_{i=1}^d$ in $\mathbb{F}$ st

$$T^{-1} A T = \begin{pmatrix} -d & & & & \\ & 1 & 2-d & & \\ & & 1 & 4-d & \\ & & & \ddots & \\ 0 & & & & 1 & d \end{pmatrix}$$

$$T^{-1} A^* T = \begin{pmatrix} d & \phi_1 & & & 0 \\ & d-2 & \phi_2 & & \\ & & d-4 & \ddots & \\ & 0 & & \ddots & \phi_d \\ & & & & -d \end{pmatrix}$$

(iii) For $0 \leq i \leq d$ let E_i (resp E_i^*) denote the
prim idempotent of A (resp A^*) for $\theta_i = d-2i$
(resp $\theta_i^* = d-2i$). Show that

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a Leonard system over $\mathbb{F}$ with param array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

Ex 102 Given a LS on V

$$\mathbb{F} = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

with param array

$$\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d \right)$$

show that the intersection numbers and dual intersection numbers satisfy

$$b_i = \varphi_{in} \frac{\gamma_i^*(\theta_i^*)}{\gamma_{in}^*(\theta_{in}^*)} \quad (0 \leq i \leq d-1)$$

$$c_i = \phi_i \frac{\gamma_{d-i}^*(\theta_i^*)}{\gamma_{d-in}^*(\theta_{in}^*)} \quad (1 \leq i \leq d)$$

$$b_i^* = \varphi_{in} \frac{\gamma_i(\theta_i)}{\gamma_{in}(\theta_{in})} \quad (0 \leq i \leq d-1)$$

$$c_i^* = \phi_{d-in} \frac{\gamma_{d-i}(\theta_i)}{\gamma_{d-in}(\theta_{in})} \quad (1 \leq i \leq d)$$

Recall for $0 \leq i \leq d$

$$\gamma_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),$$

etc.

$$\gamma_i = (\lambda - \theta_d)(\lambda - \theta_{d+1}) \cdots (\lambda - \theta_{d-i})$$

Ex 103 In the Appendix to the paper
Notes on the Leonard system classification
we see 13 examples of a parameter array.

For 3 of these examples, prove that it really
is a param array by showing PA3 - PA5.