

Math 846

Lecture 26

We continue to prove thm 82.

Recall the scalars in $\mathbb{F}$:

$$\left(\{ \theta_i \}_{i=0}^d ; \{ \theta_i^* \}_{i=0}^d ; \{ \psi_i \}_{i=1}^d ; \{ \phi_i \}_{i=1}^d \right) \quad (\star)$$

We assume $(\star)$ satisfies PA1 - PA5

We will construct a LS Φ over $\mathbb{F}$ with PA $(\star)$.

Let $V =$ vector space over $\mathbb{F}$ dim $d+1$.

Pick basis $\{u_i\}_{i=0}^d$ for V

Define $A, A^* \in \text{End}(V)$ that are represented as follows wrt $\{u_i\}_{i=0}^d$:

$$A: \begin{pmatrix} \theta_0 & \theta_1 & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & \psi_1 & & 0 \\ \theta_1^* & \psi_2 & & \\ & \ddots & & \\ 0 & & & \psi_d & \theta_d^* \end{pmatrix}$$

By construction

- A is mult free with eigenvalues $\{ \theta_i \}_{i=0}^d$
- A^* is mult free with eigenvalues $\{ \theta_i^* \}_{i=0}^d$

For $0 \leq i \leq d$ let E_i (resp E_i^*) denote the prim idempotent of A (resp A^*) for θ_i (resp θ_i^*)

The sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a Pre Leonard system on V

We will show that for $0 \leq i, j \leq d$:

$$E_i^* A E_j^* = 0 \quad \text{if} \quad i-j > 1 \quad \text{TP 1}$$

$$E_i^* A E_j^* = 0 \quad \text{if} \quad j-i > 1 \quad \text{TP 2}$$

$$E_i^* A E_j^* \neq 0 \quad \text{if} \quad i-j = 1 \quad \text{TP 3}$$

$$E_i^* A E_j^* \neq 0 \quad \text{if} \quad j-i = 1 \quad \text{TP 4}$$

$$E_i A^* E_j = 0 \quad \text{if} \quad d-i-j > 1 \quad \text{TP 5}$$

$$E_i A^* E_j = 0 \quad \text{if} \quad d=i-j > 1 \quad \text{TP 6}$$

$$E_i A^* E_j = 0 \quad \text{if} \quad j-i > 1 \quad \text{TP 7}$$

$$E_i A^* E_j \neq 0 \quad \text{if} \quad i-j = 1 \quad \text{TP 8}$$

$$E_i A^* E_j \neq 0 \quad \text{if} \quad j-i = 1 \quad \text{TP 9}$$

TP1, TP3, TP7 : By Prop 69

TP9 : By Lem 70

Note E_0^* is normalizing by Prop 69

By (PA5) $\exists \beta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ and

$\{\theta_i^*\}_{i=0}^d$ are β -rec.

$\exists \gamma \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ are (β, γ) -rec

$\exists \delta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ are (β, γ, δ) -rec.

By (PA3), for $1 \leq i \leq d$ we have

$$\psi_i = (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) = \phi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d+h}}{\theta_0 - \theta_d}$$

Denote this common value by f_i

Note

$$f_1 = \phi_1 = f_d$$

For notational convenience, define

$$f_0 = 0, \quad f_{d+1} = 0$$

TP5: By constr

$$g_i = g_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leq i \leq d+1)$$

By P88,

$$\{g_i\}_{i=0}^{d+1} \text{ is } \beta\text{-rec}$$

By this and Prop 92,

$$0 = \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right]$$

For $0 \leq i, j \leq d$ we have

$$\begin{aligned} 0 &= E_i \left(\left[A, \dots \right] \right) E_j \\ &= E_i A^* E_j (\alpha_i - \alpha_j) P(\alpha_i, \alpha_j) \end{aligned}$$

where

$$P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma (\lambda + \mu) - \delta$$

By Prop 89 (ii),

$$P(\alpha_i, \alpha_j) \neq 0 \quad \text{if } d > i-j > 1$$

so

$$E_i A^* E_j = 0 \quad \text{if } d > i-j > 1$$

✓

TP 6: Assume $d \geq 2$, else triv.

show

$$E_d A^* E_0 = 0$$

So to show

$$E_d A^* E_0 E_0^* = 0$$

by P65 and since E_0^* is normalizing.

To show

$$E_d A^* E_0 E_0^* = 0,$$

we use Prop 73 (wrap around):

$$\sum_{i=0}^{d-2} E_d A^* E_i E_0^* (\theta_i - \theta_{d-1}) = E_d E_0^* (j_1 - j_d)$$

By TP5,

$$E_d A^* E_i = 0 \quad (1 \leq i \leq d-2)$$

We saw $j_1 = j_d$

So

$$E_d A^* E_0 E_0^* (\underbrace{\theta_0 - \theta_{d-1}}_{\neq 0 \text{ since } d \geq 2}) = 0$$

$$\text{So } E_d A^* E_0 E_0^* = 0$$

$$\text{So } E_d A^* E_0 = 0 \quad \checkmark$$

TP8: We show

$$E_i A^* E_i \neq 0 \quad (i \leq d)$$

Recall $\mathbb{F} \parallel \text{PLS}$

so $\mathbb{F}^\Psi$ is PLS

Apply Prop 69 to $\mathbb{F}^\Psi$

Prop 69 (i) holds for $\mathbb{F}^\Psi$ by

TP1, TP3, TP5, TP6

so Prop 69 (iii) holds for $\mathbb{F}^\Psi$

so $\exists$ scalars $\{\varphi_i^\Psi\}_{i=1}^d$ in $\mathbb{F}$ and

a basis for V w.r.t which

$$A: \begin{pmatrix} \theta_1 & & & 0 \\ & \theta_2 & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_1^\Psi & \varphi_1^\Psi & & 0 \\ \theta_2^\Psi & \varphi_2^\Psi & & \\ & & \ddots & \\ 0 & & & \varphi_d^\Psi & \theta_d^\Psi \end{pmatrix} \quad (D)$$

We show

$$\varphi_i^\Psi = \phi_i \quad (i \leq d)$$

By (PA4),

$$\phi_i - (\theta_i^* - \theta_0^*)(\theta_{d-in} - \theta_0) = \varphi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d)$$

We will show

$$\varphi_i^{\Downarrow} - (\theta_i^* - \theta_0^*)(\theta_{d-in} - \theta_0) = \varphi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d)$$

Define

$$j_i^{\Downarrow} = \varphi_i^{\Downarrow} - (\theta_i^* - \theta_0^*)(\theta_{d-in} - \theta_0) \quad (1 \leq i \leq d)$$

and $j_0^{\Downarrow} = 0, \quad j_{d+1}^{\Downarrow} = 0$

We will show

$$j_i^{\Downarrow} = \varphi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d)$$

By Prop 88 it suff to show

$$(i) \quad j_1^{\Downarrow} = \varphi_1$$

$$(ii) \quad j_1^{\Downarrow} = j_d^{\Downarrow}$$

$$(iii) \quad \{ j_i^{\Downarrow} \}_{i=0}^{d+1} \text{ is } \beta\text{-rec}$$

() pf (i) For Φ and $\Phi^{\Downarrow}$ compute a_0
using Lem 71:

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}$$

$$a_0 = \theta_d + \frac{\varphi_1^{\Downarrow}}{\theta_d^* - \theta_1^*}$$

So

$$\varphi_1 - \varphi_1^{\Downarrow} = (\theta_1^* - \theta_0^*)(\theta_0 - \theta_d)$$

So

$$\begin{aligned} g_1^{\Downarrow} &= \varphi_1^{\Downarrow} - (\theta_1^* - \theta_0^*)(\theta_d - \theta_0) \\ &= \varphi_1 \quad \checkmark \end{aligned}$$

pf (ii)Assume $d \geq 2$, else trivApply wrap around Prop 7.3 to $\mathbb{F}^{\Downarrow}$:

$$\sum_{i=0}^{d-2} E_0 A^* E_{d-i} E_0^* (\theta_{d-i} - \theta_i) = E_0 E_0^* (j_1^{\Downarrow} - j_d^{\Downarrow})$$

In this equation, LHS = 0 since

$$E_0 A^* E_{d-i} = 0 \quad (0 \leq i \leq d-2)$$

by TP7.

Recall E_0^* is normalizing, so

$$E_0 E_0^* \neq 0$$

so

$$j_1^{\Downarrow} = j_d^{\Downarrow}$$

✓

pt (iii) Apply Prop 92 to matrices rep
 A, A^* in (B)

Recall $\{\phi_i\}_{i=0}^d$ is (β, γ, δ) -rec and
 $\{\phi_i^*\}_{i=0}^d$ is β -rec

We showed earlier that β, γ, δ rat TD1

Applying Prop 92 to the matrices rep A, A^*
 in (B) , we find

$$\{j_i^{\downarrow}\}_{i=0}^{d+1} \text{ is } \beta\text{-rec.} \quad \checkmark$$

We have shown (i) - (iii)

$$\text{So } j_i^{\downarrow} = \varphi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d)$$

$$\text{Therefore } \varphi_i^{\downarrow} = \phi_i \quad (1 \leq i \leq d)$$

$$\text{So } \varphi_i^{\downarrow} = \phi_i \neq 0 \quad (1 \leq i \leq d)$$

Applying Lem 70 to $\Phi^{\downarrow}$, get
 $E_i A^* E_{i+1} \neq 0 \quad (1 \leq i \leq d)$

TP2, TP4: Apply $3/4$ thm Prop 56:

Prop 56 (i) by TP1, TP3 ✓

→ Prop 56 (ii) → TP2, TP4

Prop 56 (iii) by TP5, TP6, TP8 ✓

Prop 56 (iv) by TP7, TP9 ✓

We have shown TP1 — TP9

So Φ is Leonard system in V

By construction:

- Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$
- Φ has dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$
- Φ has 1st split sequence $\{\varphi_i\}_{i=1}^d$

$\{\varphi_i^{\#}\}_{i=1}^d$ is 1st split sequence of $\Phi^{\#}$ $\Downarrow$

We saw $\varphi_i^{\#} = \phi_i$ ($1 \leq i \leq d$)

- Φ has 2nd split sequence $\{\phi_i\}_{i=1}^d$

By these comments,

Φ has param array $(\star)$

By Prop 81,

Φ is unique up to iso of Leonard systems

□