

Math 846

Lecture 25

Recall $\mathbb{F}$ arb, $d \in \mathbb{N}$, $V = \text{vs dim } d$.

We return our attention to a Leonard system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

By THM 38, $\exists \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ s.t.

$$0 = \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right] \quad \text{"TD1"}$$

$$0 = \left[A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A^* A + A A^*) - \delta^* A \right] \quad \text{"TD2"}$$

By THM 40, the scalars

$$\frac{\theta_{i,2} - \theta_{i,1}}{\theta_{i,1} - \theta_i},$$

$$\frac{\theta_{i,2}^* - \theta_{i,1}^*}{\theta_{i,1}^* - \theta_i^*}$$

are equal and indep of i for $2 \leq i \leq d-1$

Next goal: find out what do TD1, TD2

imply about the 1st and 2nd split sequence

$$\{\theta_i\}_{i=1}^d, \quad \{\theta_i^*\}_{i=1}^d$$

Until further notice, let

$$\{\theta_i\}_{i=0}^d; \quad \{\theta_i^v\}_{i=0}^d; \quad \{\psi_i\}_{i=1}^d$$

denote any scalars in $\mathbb{F}$.

Define $A, A^v \in \text{Mat}_{d+1}(\mathbb{F})$ by

$$A = \begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}$$

$$A^v = \begin{pmatrix} \theta_0^v & \psi_1 & & 0 \\ & \theta_1^v & \psi_2 & \\ & & \ddots & \\ 0 & & & \psi_d \\ & & & & \theta_d^v \end{pmatrix}$$

Pick any $\beta, \gamma, \delta \in \mathbb{F}$ and consider

$$[A, A^2 A^v - \beta A A^v A + A^v A^2 - \gamma (A A^v + A^v A) - \delta A^v] \quad \star$$

We find the entries in $\star$

For notational convenience, define

$$j_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad 1 \leq i \leq d$$

and $j_0 = 0, \quad j_{d+1} = 0$

LEM 90 With above notation we give the entries in $\star$

(i) the $(i+1, i-2)$ -entry is

$$\theta_{i-2}^* - (\beta_{i+1}) \theta_{i-1}^* + (\beta_{i+1}) \theta_i^* - \theta_{i+1}^*$$

$$\text{for } 2 \leq i \leq d+1$$

(ii) the $(i, i-2)$ -entry is

$$g_{i-2} - (\beta_H) g_{i-1} + (\beta_H) g_i - g_{i+1}$$

+

$$(\theta_{i-2}^x - \theta_0^x) \left(\theta_{i-2} - (\beta_H) \theta_{i-1} + (\beta_H) \theta_i - \theta_{i+1} \right)$$

+

$$(\theta_i - \theta_d) \left(\theta_{i-2}^x - (\beta_H) \theta_{i-1}^x + (\beta_H) \theta_i^x - \theta_{i+1}^x \right)$$

+

$$(\theta_{i-2}^x - \theta_i^x) \left(\theta_{i-2} - \beta \theta_{i-1} + \theta_i - \gamma \right)$$

for $2 \leq i \leq d$.

(iii) the $(i, i+1)$ -entry is

$$\varphi_{i+1} \left(\theta_{i+2} - \beta \theta_{i+1} + \theta_i - \gamma \right)$$

$$- \varphi_{i+1} \left(\theta_{i+1} - \beta \theta_i + \theta_{i+1} - \gamma \right)$$

$$+ \left(\theta_{i+1}^v - \theta_i^* \right) \left(\theta_{i+1}^2 - \beta \theta_{i+1} \theta_i + \theta_i^2 - \gamma (\theta_{i+1} + \theta_i) - \delta \right)$$

f_i is i.i.d.

(iv) the (i, i) -entry is

$$\varphi_i \left(\theta_{i+}^2 - \beta \theta_{i+} \theta_i + \theta_i^2 - \gamma (\theta_{i+} + \theta_i) - \delta \right)$$

$$- \varphi_{i+} \left(\theta_i^2 - \beta \theta_i \theta_{i+} + \theta_{i+}^2 - \gamma (\theta_i + \theta_{i+}) - \delta \right)$$

$$\text{for } 0 \leq i \leq d$$

(v) the $(i+, i)$ -entry is

$$\varphi_i (\theta_{i+} - \theta_i) \left(\theta_{i+}^2 - \beta \theta_{i+} \theta_i + \theta_i^2 - \gamma (\theta_{i+} + \theta_i) - \delta \right)$$

$$\text{for } 1 \leq i \leq d.$$

All other entries are 0.

In above formulas

$$\varphi_0 = 0,$$

$$\varphi_{d+} = 0,$$

and

$\theta_+, \theta_{d+}, \theta_{-1}^*, \theta_{d+}^*$ are indeterminates.

pf Matrix mult.

□

LEM 91 With reference to Lem 90,

assume

- $\{\theta_i\}_{i=0}^d$ are mutually distinct and (β, γ, d) -rec
- $\{\theta_i^{\gamma}\}_{i=0}^d$ are β -rec

then the $(i, i-1)$ -entry of $\star$ is

$$j_{i-2} - (\beta+1)j_{i-1} + (\beta+1)j_i - j_{i+1}$$

for $2 \leq i \leq d$. All other entries of $\star$ are 0.

pf By Lem 90.

□

(7) Prop 92 With reference to LEM 90

assume

• $\{\theta_i\}_{i=0}^d$ are mut distinct and (β, γ, d) -rec

• $\{\theta_i^*\}_{i=0}^d$ are β -rec

then

$$0 = \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right]$$

if and only if

$\{\delta_i\}_{i=0}^{d+1}$ is β -recurrent.

Pf By LEM 91

□

We are now ready to prove the classification of Leonard systems thm 82

Thm 82 (Restate) Consider a sequence of scalars in $\mathbb{F}$:

$$\left(\{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d, \{ \varphi_i \}_{i=1}^d, \{ \phi_i \}_{i=1}^d \right) \quad (\star)$$

There exists a Leonard system Φ over $\mathbb{F}$ with param array $(\star)$

if and only if the following (PA1)–(PA5) hold:

$$(PA1) \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j \quad (0 \leq i, j \leq d)$$

$$(PA2) \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d)$$

$$(PA3) \quad \text{for } 1 \leq i \leq d,$$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$$

$$(PA4) \quad \text{for } 1 \leq i \leq d$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i} - \theta_0)$$

$$(PA5) \quad \text{the scalars}$$

$$\frac{\theta_{i-2} - \theta_{i-1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i-1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and indep of i for $2 \leq i \leq d-1$.

Moreover, if Φ exists then Φ is unique up to iso

of Leonard systems.

pf thm 82.

Assume $d \geq 1$; else vacuous.Assume $\exists$ LS $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ over $\mathbb{F}$ with parameter array $\star$. We check (PA1) - (PA5)(PA1): Since A, A^* are multiples

(PA2): By Lem 70

(PA5): By thm 40

By PA5 $\exists \beta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ are β -rec.

(PA3): Define

$$f_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_i - \theta_d) \quad (1 \leq i \leq d)$$

and

$$f_0 = 0,$$

$$f_{d+1} = 0$$

We show

$$f_i = \phi_i \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d+h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d)$$

To do this, we invoke Prop 88

We will show

$$(i) \quad g_1 = \phi_1$$

$$(ii) \quad g_d = \phi_1$$

$$(iii) \quad \{g_i\}_{i=0}^{dn} \text{ is } \beta\text{-rec}$$

pf (i) : By def

$$g_1 = \phi_1 - (\theta_1^* - \theta_0^*)(\theta_0 - \theta_d)$$

By L71

$$a_0 = \theta_0 + \frac{\phi_1}{\theta_0^* - \theta_1^*}$$

By L77

$$a_0 = \theta_d + \frac{\phi_1}{\theta_0^* - \theta_1^*}$$

so

$$0 = \theta_0 - \theta_d + \frac{\phi_1 - \phi_1}{\theta_0^* - \theta_1^*}$$

so

$$(\theta_1^* - \theta_0^*)(\theta_0 - \theta_d) = \phi_1 - \phi_1$$

so

$$\phi_1 = \phi_1 - (\theta_1^* - \theta_0^*)(\theta_0 - \theta_d) = g_1 \quad \checkmark$$

pf (iii):

By def.

$$J_d = \varphi_d - (\theta_d^* - \theta_0^*)(\theta_{d+1} - \theta_d)$$

By L71

$$\theta_d^* = \theta_0^* + \frac{\varphi_d}{\theta_d - \theta_{d-1}}$$

By L77.

$$\theta_d^* = \theta_0^* + \frac{\phi_1}{\theta_d - \theta_{d-1}}$$

so

$$0 = \theta_d^* - \theta_0^* + \frac{\varphi_d - \phi_1}{\theta_d - \theta_{d-1}}$$

so

$$(\theta_d^* - \theta_0^*)(\theta_{d-1} - \theta_d) = \varphi_d - \phi_1$$

so

$$\phi_1 = \varphi_d - (\theta_d^* - \theta_0^*)(\theta_{d+1} - \theta_d) = J_d \quad \checkmark$$

pf (iii): We invoke Prop 92 $\{\theta_i\}_{i=0}^d$ is β -rec.so $\exists \gamma \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ is (β, γ) -recso $\exists \delta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ is (β, γ, δ) -rec.

By L37,

$$0 = \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* A A^* A) - \delta A^* \right]$$

By P92,

$$\{g_i\}_{i=0}^{dn} \text{ is } \beta\text{-rec} \quad \checkmark$$

(PA9): Apply PA3 to $\Phi \Downarrow$

We done in one direction

We now reverse the direction

Assume the scalars $(\star)$ satisfy (PA1)-(PA5)We will construct a LS Φ over $\mathbb{F}$ with param array $(\star)$ Let $V =$ vector space over $\mathbb{F}$ dim $d+1$ Pick a basis $\{u_i\}_{i=0}^d$ of V Define $A, A^* \in \text{End}(V)$ that are represented as follows wrt $\{u_i\}_{i=0}^d$:

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & \theta_d \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ \theta_1^* & \varphi_2 & & \\ & \ddots & & \\ 0 & & & \varphi_d \\ & & & \theta_d^* \end{pmatrix}$$