

Math 84 6

Lecture 24

Recall

$\mathbb{F}$ arb

$d \in \mathbb{N}$

$\{\theta_i\}_{i=0}^d$ is any sequence of scalars in $\mathbb{F}$

We need some formulas involving $\{\theta_i\}_{i=0}^d$

LEM 85 Assume that $\{\theta_i\}_{i=0}^d$ are mutually distinct and β -recurrent.

Pick any i, j, r, s ($0 \leq i, j, r, s \leq d$) s.t.

$$i+j = r+s, \quad r \neq s$$

Then:

(i) Suppose $\beta \neq 2, \beta \neq -2$. Then

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{q^i - q^j}{q^r - q^s}$$

where $q + q^{-1} = \beta$

(ii) Suppose $\beta = 2$ and $\text{char}(\mathbb{F}) \neq 2$. Then

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{i-j}{r-s}$$

(iii) Suppose $\beta = -2$ and $\text{char}(\mathbb{F}) \neq 2$. Then

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \begin{cases} (-1)^{i+r} \frac{i-j}{r-s} & \text{if } i+j \text{ even} \\ (-1)^{i+r} & \text{if } i+j \text{ odd} \end{cases}$$

(iv) Suppose $\beta = 0$ and $\text{char}(\mathbb{F}) = 2$. Then

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \begin{cases} 0 & \text{if } i=s \\ 1 & \text{if } i \neq s \end{cases}$$

pf Use the closed forms

(i) Obs

$$\theta_i - \theta_j = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \\ - (\alpha_1 + \alpha_2 q^j + \alpha_3 q^{-j})$$

$$= \alpha_2 (q^i - q^j) + \alpha_3 \underbrace{(q^{-i} - q^{-j})}_{q^{-i-j}(q^j - q^i)}$$

$$= (q^i - q^j)(\alpha_2 - \alpha_3 q^{-i-j})$$

So

$$\frac{\theta_i - \theta_j}{\theta_1 - \theta_2} = \frac{q^i - q^j}{q^1 - q^2} \cdot \frac{\alpha_2 - \alpha_3 q^{-i-j}}{\alpha_2 - \alpha_3 q^{-1-2}}$$

(ii) - (iv) Similar.

□

Until further notice, assume $d \geq 1$ and

$\{\theta_i\}_{i=0}^d$ are mutually distinct.

For $0 \leq i \leq d+1$ consider the sum

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (*)$$

Call this sum J_i .

Obs

$$J_0 = 0, \quad J_1 = 1, \quad J_d = 1, \quad J_{d+1} = 0$$

Also

$$J_i = J_{d-i+1} \quad 0 \leq i \leq d$$

Next goal: Find $(*)$ in closed form assuming

$\{\theta_i\}_{i=0}^d$ is β -recurrent.

To avoid trivialities assume $d \geq 3$

LEM 86

Assume $\{\theta_i\}_{i=0}^d$ are mutually distinctand β -rec. Further assume $d \geq 3$, then for $0 \leq i \leq d-1$:(i) Suppose $\beta \neq 2, \beta \neq -2$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{q^i - 1}{q - 1} \frac{q^{d-i} - 1}{q^d - 1}$$

(ii) Suppose $\beta = 2$ and $\text{char}(\mathbb{F}) \neq 2$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{i(d-i+1)}{d}$$

(iii) Suppose $\beta = -2$ and $\text{char}(\mathbb{F}) \neq 2$ and d odd. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} 0 & \text{if } i \text{ even} \\ 1 & \text{if } i \text{ odd} \end{cases}$$

(iv) Suppose $\beta = -2$ and $\text{char}(\mathbb{F}) \neq 2$ and d even. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} i/d & \text{if } i \text{ even} \\ (d-i+1)/d & \text{if } i \text{ odd} \end{cases}$$

(v) Suppose $\beta = 0$ and $\text{char}(\mathbb{F}) = 2$ and $d \geq 3$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} 0 & \text{if } i \text{ even} \\ 1 & \text{if } i \text{ odd} \end{cases}$$

pf Use LEM 85

□

LEM 87 Assume $d \geq 1$. Assume $\{\theta_i\}_{i=0}^d$ are mutually distinct and β -recurrent. Define

$$j_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leq i \leq d+1)$$

Then $\{j_i\}_{i=0}^{d+1}$ is β -recurrent.

pt Case $d=1$ vac

Case $d=2$:

$$j_0 = 0, \quad j_1 = 1, \quad j_2 = 1, \quad j_3 = 0$$

so

$$j_0 - (\beta+1)j_1 + (\beta+1)j_2 - j_3 = 0 \quad \checkmark$$

Case $d \geq 3$: check each case in LEM 86 (ex)

□

Prop 88 Let $d \geq 1$. Assume that $\{\theta_i\}_{i=0}^d$ are mutually distinct and β -recurrent.

Let $\{j_i\}_{i=0}^{d+1}$ denote any scalars in $\mathbb{F}$.

Then TFAE

$$(i) \quad j_i = j_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad 0 \leq i \leq d+1$$

(ii) the sequence $\{j_i\}_{i=0}^{d+1}$ is β -rec and

$$j_0 = 0, \quad j_1 = j_d, \quad j_{d+1} = 0$$

★

pf (i) $\rightarrow$ (ii) $\{j_i\}_{i=0}^{d+1}$ is β -rec by LEM. 87

We saw ★ above LEM 86.

(ii) $\rightarrow$ (i) Write

$$\Delta_i = j_i - j_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leq i \leq d+1)$$

$$\text{show } \Delta_i = 0 \quad (0 \leq i \leq d+1)$$

Obs

$$\Delta_0 = 0, \quad \Delta_1 = 0, \quad \Delta_d = 0, \quad \Delta_{d+1} = 0$$

For rest of pf assume $d \geq 3$; else done.

Note that $\{\Delta_i\}_{i=0}^{d+1}$ is β -rec.

We consider each closed form separately.

Case $\beta \neq 2, \beta \neq -2$

Pick $0 \neq q \in \mathbb{F}^{\text{cl}}$ s.t.

$$q + q^{-1} = \beta, \quad \exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^{\text{cl}} \text{ s.t.}$$

$$\Delta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \quad (0 \leq i \leq d+1) \quad *$$

Since $\{\Delta_i\}_{i=0}^d$ are mut dist and β -rec.

$$q^i \neq 1 \quad 1 \leq i \leq d$$

We have

$$0 = \Delta_0 = \alpha_1 + \alpha_2 + \alpha_3$$

$$0 = \Delta_1 = \alpha_1 + \alpha_2 q + \alpha_3 q^{-1}$$

$$0 = \Delta_d = \alpha_1 + \alpha_2 q^d + \alpha_3 q^{-d}$$

So

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & q & q^{-1} \\ 1 & q^d & q^{-d} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

↑

$$\det = (q-1)(q^{d+1}-1)(q^d-1)q^{-d} \neq 0$$

So

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

So

$$\Delta_i = 0 \quad (0 \leq i \leq d+1)$$

Case $\beta = 2$, $\text{char}(\mathbb{F}) \neq 2$

$\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ st

$$\Delta_i = \alpha_1 + \alpha_2 i + \alpha_3 i^2$$

$$0 \leq i \leq d-1$$

Since $\{\theta_i\}_{i=0}^d$ are mut dist and β -rec,

$\text{char}(\mathbb{F}) = 0$ or $\text{char}(\mathbb{F}) > d$

$$0 = \Delta_0 = \alpha_1$$

$$0 = \Delta_1 = \alpha_1 + \alpha_2 + \alpha_3$$

$$0 = \Delta_d = \alpha_1 + \alpha_2 d + \alpha_3 d^2$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & d & d^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\uparrow$$

$$\det = d(d-1) \neq 0$$

$$\text{So } \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$\text{So } \Delta_i = 0 \quad (0 \leq i \leq d-1)$$

Case $\beta = -2$, $\text{char}(\mathbb{F}) \neq 2$

$\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ st

$$\Delta_i = \alpha_1 + \alpha_2(-1)^i + \alpha_3 i(-1)^i \quad (0 \leq i \leq d-1)$$

Since $\{\theta_i\}_{i=0}^d$ are mult dist and β -rec,

$$\text{char}(\mathbb{F}) = 0 \quad \text{or} \quad 2 \text{char}(\mathbb{F}) > d$$

$$0 = \Delta_0 = \alpha_1 + \alpha_2$$

$$0 = \Delta_1 = \alpha_1 - \alpha_2 - \alpha_3$$

$$0 = \Delta_d = \alpha_1 + \alpha_2(-1)^d + \alpha_3 d(-1)^d$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & (-1)^d & d(-1)^d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\uparrow$$

$$\det = \begin{cases} -2d & \text{if } d \text{ even} \\ 2(d-1) & \text{if } d \text{ odd} \end{cases}$$

$$\text{For } d \text{ even} \quad -2d = \begin{matrix} -2 & 2 & \frac{d}{2} \\ \neq & \neq & \neq \\ 0 & 0 & 0 \end{matrix} \neq 0$$

$$\text{For } d \text{ odd} \quad 2(d-1) = \begin{matrix} 2 & 2 & \frac{d-1}{2} \\ \neq & \neq & \neq \\ 0 & 0 & 0 \end{matrix} \neq 0$$

So $\alpha_1 = \alpha_2 = \alpha_3 \neq 0$. So $\Delta_i \neq 0$ ($0 \leq i \leq d-1$)

Case $\beta = 0$, $\text{char}(\mathbb{F}) = 2$, $d = 3$

$$\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F} \text{ st}$$

$$\Delta_i = \alpha_1 + \alpha_2 i + \alpha_3 \begin{pmatrix} i \\ i \\ i \end{pmatrix} \quad 0 \leq i \leq 4$$

$$0 = \Delta_0 = \alpha_1$$

$$0 = \Delta_1 = \alpha_1 + \alpha_2$$

$$0 = \Delta_3 = \alpha_1 + \alpha_2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\det = 1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$\Delta_i = 0 \quad (0 \leq i \leq 4)$$

□

Assume $\{\theta_i\}_{i=0}^d$ are mutually distinct and

$$(\beta, \gamma, \delta) - \text{rec}$$

Recall the polynomial

$$P(x, y) = x^2 - \beta xy + y^2 - \gamma(x+y) - \delta$$

Obs $(\beta, \gamma, \delta) - \text{rec}$ means

$$P(\theta_i, \theta_j) = 0 \quad 1 \leq i, j \leq d$$

Prop 89 With above notation and assumptions

$$(i) \quad P(x, \theta_j) = (x - \theta_{j-1})(x - \theta_{j+1}) \quad (1 \leq j \leq d-1)$$

$$(ii) \quad \text{For } 0 \leq i, j \leq d,$$

$$P(\theta_i, \theta_j) = 0 \text{ implies } |i-j| = 1 \text{ or } i, j \in \{0, d\}$$

pf (i) $P(x, \theta_j)$ is monic poly in x with degree 2.

$$P(\theta_{j-1}, \theta_j) = 0 \quad \text{and} \quad P(\theta_{j+1}, \theta_j) = 0$$

So $\theta_{j-1}, \theta_{j+1}$ are roots of $P(x, \theta_j)$ ✓

(ii) Assume $P(\theta_i, \theta_j) = 0$

WLOG $1 \leq i \leq d-1$ or $1 \leq j \leq d-1$

else $i, j \in \{a, d\}$,

Swapping i, j if nec, WLOG $1 \leq j \leq d-1$

By (i)

$$0 = P(\theta_i, \theta_j) = (\theta_i - \theta_{j-1})(\theta_i - \theta_{j+1})$$

So $i = j-1$ or $i = j+1$

So $|i-j| = 1$

□