

Math 846

Lecture 23

Until further notice, we fix a

Leonard system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Recall the eigenvalue sequence $\{\theta_i\}_{i=0}^d$
 ... dual ... $\{\theta_i^*\}_{i=0}^d$

DEF 74 By the first split sequence of Φ

we mean the sequence $\{\varphi_i\}_{i=1}^d$ from Prop 69

Let $\{\phi_i\}_{i=1}^d$ denote the 1st split sequence

of $\Phi^{\vee}$, we call $\{\phi_i\}_{i=1}^d$ the second

split sequence of Φ

By Lem 70

$$\varphi_i \neq 0$$

$$\phi_i \neq 0$$

$$(1 \leq i \leq d)$$

(1) LEM 75 $\exists$ basis for V w.r.t which

$$A: \begin{pmatrix} \theta_d & & & & 0 \\ & 1 & \theta_{d-1} & & \\ & & 1 & & \\ 0 & & & \ddots & \\ & 0 & & & 1 & \theta_0 \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & & & 0 \\ & \theta_1^* & \phi_2 & & \\ & & \ddots & \ddots & \\ 0 & & & & \phi_d \\ & & & & \theta_d^* \end{pmatrix}$$

p.f. Apply Prop 69 to $\underline{F}^{\vee}$

() LEM 76 For a LS Φ' over $\mathbb{F}$. TFAE

(i) Φ, Φ' are isomorphic

(ii) Φ, Φ' have same

equal rep, dual equal rep, 1st split rep.

(iii) Φ, Φ' have same

equal rep, dual equal rep, 2nd split rep.

pf (i) $\Leftrightarrow$ (ii) By Prop 69
 (i) $\Leftrightarrow$ (iii) By LEM 75

□

(i) LEM 77 For $d \geq 1$,

$$(\text{i}) \quad a_0 = \theta_d + \frac{\phi_1}{\theta_0^* - \theta_1^*}$$

$$a_i = \theta_{d-i} + \frac{\phi_i}{\theta_i^* - \theta_{i+1}^*} + \frac{\phi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (1 \leq i \leq d-1)$$

$$a_d = \theta_0 + \frac{\phi_d}{\theta_d^* - \theta_{d+1}^*}$$

(ii)

$$a_0^* = \theta_d^* + \frac{\phi_d}{\theta_0 - \theta_1}$$

$$a_i^* = \theta_{d-i}^* + \frac{\phi_{d-i}}{\theta_i - \theta_{i+1}} + \frac{\phi_{d-i+1}}{\theta_i - \theta_{i+1}} \quad (1 \leq i \leq d-1)$$

$$a_d^* = \theta_0^* + \frac{\phi_1}{\theta_d - \theta_{d+1}}$$

Pf Apply LEM 77 to Φ^V .

LEM 78 For $1 \leq i \leq d$ the scalar ϕ_i is equal to each of

$$(\theta_i^y - \theta_{i-1}^x) \left(\theta_d + \dots + \theta_{d-i+1} - a_0 - \dots - a_{i-1} \right),$$

$$(\theta_{i-1}^y - \theta_i^x) \left(\theta_0 + \dots + \theta_{d-i} - a_d - \dots - a_i \right),$$

$$(\theta_{d-i} - \theta_{d-i+1}) \left(\theta_0^x + \dots + \theta_{i-1}^y - a_d^y - \dots - a_{d-i+1}^y \right),$$

$$(\theta_{d-i+1} - \theta_{d-i}) \left(\theta_d^x + \dots + \theta_i^y - a_0^x - \dots - a_{d-i}^x \right).$$

p f Backsolve using LEM 77.

□

DEF 79 By the parameter array of Φ

we mean the sequence

$$\left(\underbrace{\{\theta_i\}_{i=0}^d}_{\text{eigenvalue sequence}} ; \underbrace{\{\theta_i^*\}_{i=0}^d}_{\text{dual eigenvalue sequence}} ; \underbrace{\{\varphi_i\}_{i=1}^d}_{\text{1st split sequence}} ; \underbrace{\{\phi_i\}_{i=1}^d}_{\text{2nd split sequence}} \right)$$

LEM 80 The parameter arrays of

$$\Phi, \Phi^\Psi, \Phi^\downarrow, \Phi^*$$

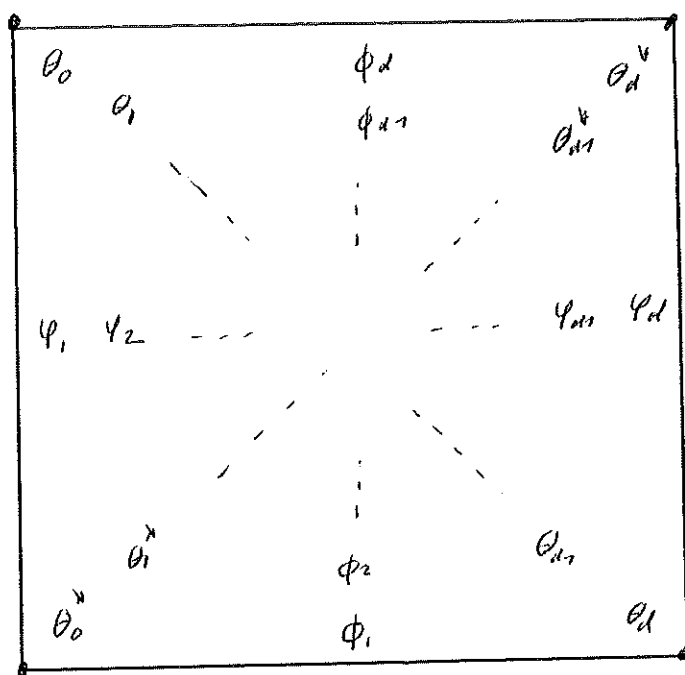
are related as follows.

LS	Parameter array
Φ	$(\{\theta_i\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^d ; \{\varphi_i\}_{i=1}^d ; \{\phi_i\}_{i=1}^d)$
Φ^Ψ	$(\{\theta_{d-i}\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^d ; \{\phi_i\}_{i=1}^d ; \{\varphi_i\}_{i=1}^d)$
$\Phi^\downarrow$	$(\{\theta_i\}_{i=0}^d ; \{\theta_{d-i}^*\}_{i=0}^d ; \{\phi_{d-i+1}\}_{i=1}^d ; \{\varphi_{d-i+1}\}_{i=1}^d)$
Φ^*	$(\{\theta_i^*\}_{i=0}^d ; \{\theta_i\}_{i=0}^d ; \{\varphi_i\}_{i=1}^d ; \{\phi_{d-i+1}\}_{i=1}^d)$

pf Compare Lem 72, 78 using table above Lem 53

□

The group D_4 acts as follows



D_4 element

Line of reflection (mirror)



Prop 81 Two Leonard systems over $\mathbb{F}$
are isomorphic iff they have the same
parameter array.

pf By LEM 76.

□

We now state the classification of Leonard systems.

F arb

Theorem 81 Consider a sequence of scalars in F :

$$\left(\{ \theta_i \}_{i=0}^d ; \{ \theta_i^* \}_{i=0}^d ; \{ \varphi_i \}_{i=1}^d ; \{ \phi_i \}_{i=1}^d \right) \quad (\star)$$

There exists a Leonard system Φ over F with parameter array $(\star)$ if and only if the following (PA1) — (PA5) hold:

$$(PA1) \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \text{ if } i \neq j \quad (0 \leq i, j \leq d)$$

$$(PA2) \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d)$$

$$(PA3) \quad \text{For } 1 \leq i \leq d$$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$$

$$(PA4) \quad \text{For } 1 \leq i \leq d$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$$

$$(PA5) \quad \text{The scalars}$$

$$\frac{\theta_{i-2} - \theta_{d-i}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{d-i}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and indep of i for $2 \leq i \leq d-1$. Moreover if Φ exists then Φ is unique up to iso of Leonard systems.

We will prove the above thm shortly. After that we will obtain the explicit sols to (A1)-(A5) in closed form. There are 13 cases.

DEF 83 By a parameter array over $\mathbb{F}$ of diameter d

we mean a sequence $(\star)$ of scalars in $\mathbb{F}$

that satisfy (PA1) - (PA5).

thm 82 gives a bijection between the following two sets:

- (i) the parameter arrays over $\mathbb{F}$ that have diam d
- (ii) the iso classes of Leonard systems over $\mathbb{F}$ that have diameter d

Note For $d \geq 1$, a parameter array

$$\left(\{ \theta_i \}_{i=0}^d ; \{ \theta_i^* \}_{i=0}^d ; \{ \varphi_i \}_{i=1}^d ; \{ \phi_i \}_{i=1}^d \right)$$

is uniquely determined by

$$\varphi_1 ; \{ \theta_i \}_{i=0}^d ; \{ \theta_i^b \}_{i=0}^d.$$

Aside: Some formulas about recurrent sequences.

Let $\{a_i\}_{i=0}^d$ denote any sequence of scalars in $\mathbb{F}$.

Earlier we described the related concepts:

- recurrent
- β -recurrent
- (β, γ) -recurrent
- (β, γ, δ) -recurrent

Recall the closed forms for recurrent sequences

LEM 84 Assume that $\{\theta_i\}_{i=0}^d$ are mutually distinct and β -rec. Then:

(i) Suppose $\beta \neq 2$, $\beta \neq -2$, and pick $0 \neq q \in \mathbb{F}^{\text{al}}$ st $\beta = q + q^{-1}$. Then

$$q^i \neq 1 \quad (1 \leq i \leq d)$$

(ii) Suppose $\beta = 2$ and $\text{char}(\mathbb{F}) = p$, $p \geq 3$. Then

$$d < p.$$

(iii) Suppose $\beta = -2$ and $\text{char}(\mathbb{F}) = p$, $p \geq 3$. Then

$$d < 2p.$$

(iv) Suppose $\beta = 0$ and $\text{char}(\mathbb{F}) = 2$. Then

$$d \leq 3.$$

pf We use the closed forms for $\{\theta_i\}_{i=0}^d$

(i) $\forall n \ 0 \leq i \leq d$

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}$$

$\forall n \ 1 \leq i \leq d$

$$0 \neq \theta_i - \theta_0$$

$$= \alpha_2 (q^i - 1) + \alpha_3 (q^{-i} - 1)$$

$$= (q^i - 1)(\alpha_2 - q^{-i} \alpha_3)$$

$\neq 0$

(ii) $\forall n \ 0 \leq i \leq d$

$$\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 i^2$$

Suppose $d \geq p$.

$$0 \neq \theta_p - \theta_0 = \alpha_1 + \alpha_2 \underset{0}{p} + \alpha_3 \underset{0}{p^2} - \alpha_1$$

$$= 0 \quad \text{cont}$$

(iii) $\forall n \ 0 \leq i \leq d$

$$\theta_i = \alpha_1 + \alpha_2 (-1)^i + \alpha_3 i(-1)^i$$

Suppose $d \geq 2p$

$$0 \neq \theta_{2p} - \theta_0$$

$$= \alpha_1 + \alpha_2 + \underbrace{\alpha_3}_{=0} \binom{2p}{2} - \alpha_1 - \alpha_2$$

$$= 0 \quad \text{cont.}$$

(iv) For $0 \leq i \leq d$

$$\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 \binom{i}{2}$$

Suppose $d \geq 4$

$$0 \neq \theta_4 - \theta_0$$

$$= \alpha_1 + \underbrace{\alpha_2}_{=0} 4 + \underbrace{\alpha_3}_{=0} \binom{4}{2} - \alpha_1$$

$$= 0 \quad \text{cont.}$$

i	$\binom{i}{2} \pmod{2}$
0	0
1	0
2	1
3	1
4	0
5	0
6	1
7	1
8	0
$\vdots$	$\vdots$

□