

Math 846

Lecture 22

$\mathbb{F}$ arb, $d \in \mathbb{N}$, V vector space dim dH .

Given PLS on V :

$$\mathbb{F} = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Assume $\mathbb{F}$ satisfies (i)-(iii) in Prop 69

WLOG

$$A = \begin{pmatrix} \theta_0 & & & 0 \\ 1 & a_1 & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix}$$

$$A^* = \begin{pmatrix} \theta_0^* & \psi_1 & & 0 \\ \theta_1^* & \psi_2 & & \\ & \ddots & & \\ 0 & & & \psi_d \\ & & & \theta_d^* \end{pmatrix}$$

Next goal: find

$$a_i \quad a_i^* \quad 0 \leq i \leq d$$

Recall

$$a_i = \kappa(AE_i^*)$$

$$a_i^* = \kappa(A^*E_i)$$

Case $d=0$

$$A = (\theta_0)$$

$$A^* = (\theta_0^*)$$

$$E_0 = (1)$$

$$E_0^* = (1)$$

$$a_0 = \theta_0$$

$$a_0^* = \theta_0^*$$

(1) LEM 71 F_n $d \geq 1$

$$(i) \quad a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^y - \theta_1^y}$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^y - \theta_{i+1}^y} + \frac{\varphi_{i+1}}{\theta_i^y - \theta_{i+1}^y} \quad (1 \leq i \leq d-1)$$

$$a_d = \theta_d + \frac{\varphi_d}{\theta_d^y - \theta_{d+1}^y}$$

(ii)

$$a_0^* = \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i+1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \quad (1 \leq i \leq d-1)$$

$$a_d^* = \theta_d^* + \frac{\varphi_d}{\theta_d - \theta_{d+1}}$$

pf (i) For $0 \leq r \leq d$ compute

$$\text{tr}(AA^{*r})$$

in 2 ways.

View I

Use

$$A^{*r} = \sum_{i=0}^d a_i^{*r} E_i^{*r}$$

$$\text{tr}(AA^{*r}) = \sum_{i=0}^d a_i^{*r} \underbrace{\text{tr}(AE_i^{*r})}_{= a_i}$$

$$= \sum_{i=0}^d a_i a_i^{*r}$$

1) New II Use matrix forms

$$\text{tr}(AA^{*r}) = \sum_{i=0}^d (AA^{*r})_{i,i}$$

$$= \sum_{i=0}^d \theta_i (\theta_i^{*r}) + \sum_{i=1}^d \underbrace{A_{i,i-1}}_{=1} (A^{*r})_{i-1,i}$$

$$\left[\begin{aligned} (A^{*r})_{i-1,i} &= \varphi_i \left(\theta_i^{*r-1} + \theta_i^* \theta_i^{*r-2} + \dots + \theta_i^{*r-1} \right) \\ &= \varphi_i \frac{\theta_{i-1}^{*r} - \theta_i^{*r}}{\theta_{i-1}^* - \theta_i^*} \end{aligned} \right]$$

$$= \sum_{i=0}^d \theta_i (\theta_i^{*r}) + \sum_{i=1}^d \varphi_i \frac{\theta_{i-1}^{*r} - \theta_i^{*r}}{\theta_{i-1}^* - \theta_i^*}$$

(i) By I, II

$$0 = \left(\theta_0 + \frac{\varphi_1}{\theta_0^y - \theta_1^x} - a_0 \right) \theta_0^{*r}$$

+

$$\sum_{i=1}^{d-1} \left(\theta_i + \frac{\varphi_i}{\theta_i^y - \theta_{i+1}^x} + \frac{\varphi_{i+1}}{\theta_i^y - \theta_{i+1}^y} - a_i \right) \theta_i^{*r}$$

+

$$\left(\theta_d + \frac{\varphi_d}{\theta_d^y - \theta_{d-1}^y} - a_d \right) \theta_d^{*r}$$

The above holds for $0 \leq r \leq d$.

The scalars $\{\theta_i^x\}_{i=0}^d$ are mutually distinct.

It follows that each term () is 0.

(ii) Compute $\text{tr}(A^*A^r)$ in 2 ways using similar arguments. □

LEM 72 For fixed the scalar φ_i is equal to each of:

$$(\theta_i^x - \theta_{i-1}^x) \left(\theta_0 + \theta_1 + \dots + \theta_{i-1} - a_0 - a_1 - \dots - a_{i-1} \right),$$

$$(\theta_i^y - \theta_{i-1}^y) \left(\theta_1 + \theta_2 + \dots + \theta_d - a_i - a_{i+1} - \dots - a_d \right),$$

$$(\theta_i - \theta_{i-1}) \left(\theta_0^y + \theta_1^y + \dots + \theta_{i-1}^y - a_0^y - a_1^y - \dots - a_{i-1}^y \right),$$

$$(\theta_{i-1} - \theta_i) \left(\theta_i^y + \theta_{i+1}^y + \dots + \theta_d^y - a_i^y - a_{i+1}^y - \dots - a_d^y \right).$$

pf Backsolve using LEM 71 and

$$a_0 + \dots + a_d = \theta_0 + \dots + \theta_d,$$

$$a_0^y + \dots + a_d^y = \theta_0^y + \dots + \theta_d^y$$

For notational convenience, define

$$f_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{in} - \theta_0) \quad 1 \leq i \leq d$$

and

$$f_0 = 0, \quad f_{dn} = 0$$

Prop 73 For $d \geq 2$

$$\sum_{i=0}^{d-2} E_d A^* E_i E_0^* (\theta_i - \theta_{d-1}) = E_d E_0^* (\theta_1 - \theta_d)$$

"wrap around result"

pf obs

$$\sum_{i=0}^d E_d A^* E_i E_0^* (\theta_i - \theta_{d-1})$$

$$= E_d A^* A E_0^* - \theta_{d-1} E_d A^* E_0^*$$

$$= E_d A^* A E_0^* - \theta_{d-1} \theta_0^* E_d E_0^*$$

obs

$$E_d A^* E_d E_0^* (\theta_d - \theta_{d-1})$$

$$= a_d^* (\theta_d - \theta_{d-1}) \bar{E}_d E_0^*$$

obs

$$E_d A^* A E_0^* = \sum_{l=0}^d E_d A^* E_l^* A E_0^*$$

$$= \sum_{l=0}^d \theta_l^* E_d \underbrace{E_l^* A E_0^*}_{\substack{= \\ 0 \text{ if } l \geq 2}}$$

$$= \sum_{l=0}^1 \theta_l^* E_d E_l^* A E_0^*$$

$$= \theta_0^* E_d \underbrace{E_0^* A E_0^*}_{= a_0 E_0^*} + \theta_1^* E_d E_1^* A E_0^*$$

$$= a_0 \theta_0^* E_d E_0^* + \theta_1^* E_d E_1^* A E_0^*$$

Find $E_d E_1^* A E_0^*$

$$E_d A \bar{E}_0^\vee = \sum_{l=0}^d E_d \bar{E}_l^\vee A \bar{E}_0^\vee$$

$$\parallel$$

$$\theta_d E_d \bar{E}_0^\vee = \sum_{l=0}^1 E_d \bar{E}_l^\vee A \bar{E}_0^\vee$$

$$= a_0 E_d \bar{E}_0^\vee + E_d \bar{E}_1^\vee A \bar{E}_0^\vee$$

$$\text{so } E_d \bar{E}_1^\vee A \bar{E}_0^\vee = (\theta_d - a_0) E_d \bar{E}_0^\vee$$

obs

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$$\sum_{i=0}^{d-1} E_d A^* E_i E_0^* (\theta_i - \theta_{d-1})$$

$$= \underbrace{\sum_{i=0}^d E_d A^* E_i E_0^* (\theta_i - \theta_{d-1})}_{\parallel} - \underbrace{E_d A^* E_d E_0^* (\theta_d - \theta_{d-1})}_{\parallel}$$

$$\underbrace{E_d A^* A E_0^*}_{\parallel} - \theta_{d-1} \theta_0^* E_d E_0^*$$

$$a_d^* (\theta_d - \theta_{d-1}) E_d E_0^*$$

$$a_0 \theta_0^* E_d E_0^* + \theta_1^* (\theta_d - a_0) E_d E_0^*$$

$$= E_d E_0^* \left(\underbrace{a_0 (\theta_0^* - \theta_1^*)}_{\parallel} + \theta_d \theta_1^* - \theta_{d-1} \theta_0^* - \underbrace{a_d^* (\theta_d - \theta_{d-1})}_{\parallel} \right)$$

$$\theta_0 (\theta_0^* - \theta_1^*) + \underbrace{\varphi_1}_{\parallel}$$

$$j_1 + (\theta_1^* - \theta_0^*) (\theta_0 - \theta_d)$$

$$- \theta_d^* (\theta_d - \theta_{d-1}) - \underbrace{\varphi_d}_{\parallel}$$

$$j_d + (\theta_d^* - \theta_{d-1}^*) (\theta_{d-1} - \theta_d)$$

$$= E_d E_0^* (j_1 - j_d) \quad \checkmark$$

cancellation

	θ_0^x	θ_1^x	θ_d^x
θ_0	$\begin{matrix} 1 \\ -1 \end{matrix}$	$\begin{matrix} -1 \\ 1 \end{matrix}$	
θ_{d1}	$\begin{matrix} -1 \\ 1 \end{matrix}$		$\begin{matrix} 1 \\ -1 \end{matrix}$
θ_d	$\begin{matrix} 1 \\ -1 \end{matrix}$	$\begin{matrix} -1 \\ 1 \end{matrix}$	$\begin{matrix} -1 \\ 1 \end{matrix}$

