

Math 846

Lecture 21

Recall

$\mathbb{F}$ arb

Fix $d \in \mathbb{N}$

vs V dim $d+1$

Pre Leonard system Φ on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^\vee; \{E_i^\vee\}_{i=0}^d)$$

We continue to prove Lem 68...

(ii) $\rightarrow$ (iii) obs

$$\sum_{j=0}^i E_j^* V = \sum_{j=0}^i p_j(A) E_0^* V \stackrel{\leftarrow \deg j}{=} \sum_{j=0}^i A^j E_0^* V$$

$$\Uparrow$$

 $\dim = i+1$

$$\Uparrow$$

 $\dim \leq i+1$

Must have = everywhere.

(iii) $\rightarrow$ (i) $\forall n \ 0 \leq i \leq d$ def

$$W_i = E_0^* V + \dots + E_i^* V = E_0^* V + A E_0^* V + \dots + A^i E_0^* V$$

 $\forall n \ 0 \leq i, j \leq d$

$$E_i^* W_j = \begin{cases} 0 & \text{if } i > j \\ \neq 0 & \text{if } i \leq j \end{cases}$$

 $\forall n \ 0 \leq j \leq d-1$

$$W_j + A W_j = W_{j+1}$$

• For $0 \leq i, j \leq d$ st $i-j > 1$

show

$$E_i^* A E_j^* = 0$$

obs

$$E_i^* A E_j^* V \leq E_i^* A W_j$$

$$\leq E_i^* (A W_j + W_j)$$

$$= E_i^* (W_{j+1})$$

$$= 0$$

since $i > j+1$

• For $0 \leq i, j \leq d$ st $i-j = 1$

show

$$E_i^* A E_j^* \neq 0$$

obs

$$0 \neq E_i^* W_i = E_i^* (W_j + A W_j)$$

$$= E_i^* A W_j = E_i^* A (E_0^* V + \dots + E_j^* V)$$

$$= E_i^* A E_j^* V \quad \checkmark$$

Assume (i)-(iii) Set $i=d$ in (iii) to get $V = \bigoplus E_0^* V$
 So E_0^* is Normalizing

□

Prop 69 TFAE:

(i) $\text{fn } 0 \leq i, j \leq d,$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

$$E_i A^* E_j = 0 \text{ if } j-i > 1$$

(ii) $\exists$ decomp $\{U_i\}_{i=0}^d$ of V st

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leq i \leq d-1), \quad (A - \theta_d I)U_d = 0$$

$$(A^* - \theta_i^* I)U_i \subseteq U_{i+1} \quad (1 \leq i \leq d), \quad (A^* - \theta_0^* I)U_0 = 0$$

(iii) $\exists \varphi_i \in \mathbb{F} \quad (1 \leq i \leq d)$ and a basis $\text{fn } V$ wrt which

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & & \ddots & \\ 0 & & & \ddots & \\ & & & & \theta_d \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ & \theta_1^* & \varphi_2 & \\ & & \ddots & \\ 0 & & & \varphi_d & \\ & & & & \theta_d^* \end{pmatrix}$$

Assume (i)-(iii). Then E_0^* is normalizing and

$$U_i = \tau_i(A) E_0^* V \quad (0 \leq i \leq d)$$

the basis in (iii) is

$$\{\tau_i(A)v\}_{i=0}^d$$

$$0 \neq v \in E_0^* V$$

this basis induces $\{U_i\}_{i=0}^d$.

() Pf (i) $\rightarrow$ (ii) obs E_0^* is normalizing

Define

$$U_i = T_i(A) E_0^* V \quad (0 \leq i \leq d)$$

Then

$$\bullet \quad \{U_i\}_{i=0}^d \text{ is a decomp of } V$$

$$\bullet \quad (A - \alpha_i I) U_i = U_{i+1} \quad (0 \leq i \leq d-1)$$

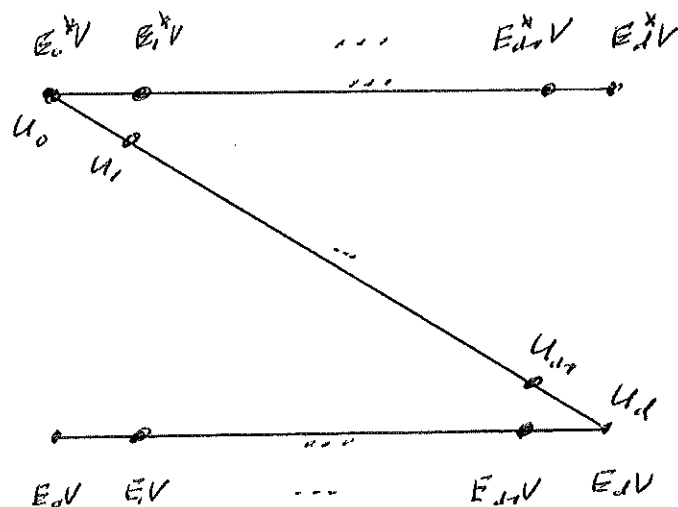
$$\bullet \quad (A - \alpha_d I) U_d = 0$$

$$\bullet \quad U_0 + U_1 + \dots + U_d = E_0 V + E_1 V + \dots + E_d V$$

$0 \leq i \leq d$

$$\bullet \quad U_0 + U_1 + \dots + U_i = E_0^* V + A E_0^* V + \dots + A^i E_0^* V$$

$$= E_0^* V + E_1^* V + \dots + E_i^* V$$



$\bar{F}_n \quad 1 \leq i \leq d$

$$\begin{aligned}
 (A^x - \theta_i^x I) u_i &\leq (A^x - \theta_i^x I) (u_0 + \dots + u_i) \\
 &= (A^x - \theta_i^x I) (\bar{E}_0^x V + \dots + \bar{E}_i^x V) \\
 &= \bar{E}_0^x V + \dots + \bar{E}_{i-1}^x V \\
 &= u_0 + \dots + u_{i-1}
 \end{aligned}$$

Also

$$\begin{aligned}
 (A^x - \theta_i^x I) u_i &\leq (A^x - \theta_i^x I) (u_0 + \dots + u_d) \\
 &= (A^x - \theta_i^x I) (\bar{E}_0^x V + \dots + \bar{E}_d^x V) \\
 &\leq \bar{E}_0^x V + \dots + \bar{E}_d^x V \\
 &= u_0 + \dots + u_d
 \end{aligned}$$

So

$$\begin{aligned}
 (A^x - \theta_i^x I) u_i &\leq (u_0 + \dots + u_{i-1}) \wedge (u_{i+1} + \dots + u_d) \\
 &= u_i
 \end{aligned}$$

Also

$$\begin{aligned}
 (A^x - \theta_0^x I) u_0 &= (A^x - \theta_0^x I) \bar{E}_0^x V \\
 &= 0
 \end{aligned}$$

(iii) $\rightarrow$ (i) For $0 \leq i \leq d$ show

$$u_0 + \dots + u_i = E_0^* V + \dots + E_i^* V$$

obs

$$\begin{aligned} u_0 + \dots + u_i &\subseteq \ker \left((A^* - \theta_0^* I) \dots (A^* - \theta_i^* I) \right) \\ &= E_0^* V + \dots + E_i^* V \end{aligned}$$

LHS, RHS have dim $i+1$

✓

For $0 \leq i \leq d$ show

$$u_i + \dots + u_d = E_i V + \dots + E_d V$$

obs

$$\begin{aligned} u_i + \dots + u_d &\subseteq \ker \left((A - \theta_i I) \dots (A - \theta_d I) \right) \\ &= E_i V + \dots + E_d V \end{aligned}$$

LHS, RHS dim $d-i+1$

✓

By emk

$$U_i = T_i(A) E_0^* V$$

$0 \leq i \leq d$

For $0 \leq i, j \leq d$ show

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

★

Apply LEM 68

For $0 \leq i \leq d$

$$\begin{aligned} E_0^* V + \dots + E_i^* V &= U_0 + \dots + U_i \\ &= T_0(A) E_0^* V + \dots + T_i(A) E_0^* V \\ &= E_0^* V + A E_0^* V + \dots + A^i E_0^* V \end{aligned}$$

So ★ follows from LEM 68.

For $0 \leq i, j \leq d$ s.t. $j-i > 1$ show

$$E_i A^* E_j = 0$$

obs

$$E_i A^* E_j V \subseteq E_i A^* (E_j V + \dots + E_d V)$$

$$= E_i A^* (u_j + \dots + u_d)$$

$$\subseteq E_i (u_{j-1} + u_j + \dots + u_d)$$

$$= E_i (E_{j-1} V + \dots + E_d V)$$

$$= 0 \quad \text{since } i < j-1$$

(i') $\Leftrightarrow$ (iii') Assertion (iii') is reformulation of
Assertion (i') in terms of matrices.

Assume (i) $\rightarrow$ (iii') Last assertions are clear

$\square$

LEM 70 Assume the equiv conditions (i)-(iii)
hold in Prop 69

then for $1 \leq i \leq d$ TFAE

$$(i) \quad E_{i \rightarrow} A^* E_i \neq 0$$

$$(ii) \quad (A^* - \theta_i^* I) U_i = U_{i \rightarrow}$$

$$(iii) \quad \psi_i \neq 0$$

pf (i) $\rightarrow$ (ii) Suppose not. Then

$$(A^* - \theta_i^* I) U_i = 0$$

$$\text{so } A^* U_i \leq U_i$$

obs

$$\begin{aligned} 0 \neq E_{i \rightarrow} A^* E_i V &\leq E_{i \rightarrow} A^* (E_i V + \dots + E_d V) \\ &= E_{i \rightarrow} A^* (U_i + \dots + U_d) \\ &\leq E_{i \rightarrow} (U_i + \dots + U_d) \\ &= E_{i \rightarrow} (E_i V + \dots + E_d V) \\ &= 0 \quad \text{cont} \end{aligned}$$

(ii) $\rightarrow$ (i) Suppose $E_i A^* E_i = 0$

obs

$$u_i = (A^* - \theta_i^* I) u_i$$

$$\subseteq (A^* - \theta_i^* I) (u_i + t + u_d)$$

$$= (A^* - \theta_i^* I) (E_i V + t + E_d V)$$

$$\subseteq E_i V + t + E_d V$$

$$= u_i + t + u_d$$

But $\{u_i\}_{i=0}^d$ is decomp, cont.

(ii) $\Leftarrow$ (iii) Use the matrix rep of A^* in Prop 69

□