

Math 846

Lecture 20

Recall:

$\mathbb{F}$ arb

Fix $d \in \mathbb{N}$

vs V dim $d+1$

Pre Leonard system on V :

$$\underline{\Phi} = (A; \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

Prop 56 Given PLS on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Consider 4 conditions:

$$(i) \quad \text{Fn } 0 \leq i, j \leq d \\ E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

$$(ii) \quad \text{Fn } 0 \leq i, j \leq d \\ E_i^* A E_j^* = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases}$$

$$(iii) \quad \text{Fn } 0 \leq i, j \leq d \\ E_i A^* E_j = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

$$(iv) \quad \text{Fn } 0 \leq i, j \leq d \\ E_i A^* E_j = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases}$$

Assume that at least 3 of (i) - (iv) hold. Then each of (i) - (iv) hold. In other words,

Φ is a LS on V .

" 3/4 Theorem "

pf Swapping $A \leftrightarrow A^*$ if nec. WLOG

(i), (ii) hold.

Now $\mathbb{F}$ satisfies $\star$

By Lem 55 $\exists$ invariant $\tau \in \text{End}(V)$

that fixes A, A^* .

For $0 \leq i, j \leq d$.

$$\begin{aligned} (E_i A^* E_j)^{\tau} &= E_j^{\tau} (A^*)^{\tau} E_i^{\tau} \\ &= E_j A^* E_i \end{aligned}$$

So $E_i A^* E_j = 0$ iff $E_j A^* E_i = 0$

So (iii) holds iff (iv) holds.

The result follows.

□

We continue to discuss the PLS

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

on V ,

let $\lambda =$ indeterminate

Recall polynomial algebra $\mathbb{F}[\lambda]$

DEF 57 For $0 \leq i \leq d$ define

$$\tau_i, \eta_i, \tau_i^*, \eta_i^* \in \mathbb{F}[\lambda]$$

by

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1})$$

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1})$$

$$\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*)$$

$$\eta_i^* = (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*)$$

Each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic degree i

We mention some results about τ_i, η_i

Similar results apply to τ_i^*, η_i^* .

LEM 58 The vectors $\{\tau_i(A)\}_{i=0}^d$ form a basis for $\mathcal{D}$
and the vectors $\{\eta_i(A)\}_{i=0}^d$ form a basis for $\mathcal{D}$.

pf Since $\{A_i\}_{i=0}^d$ is a basis for $\mathcal{D}$, and
 τ_i, η_i has degree i for $0 \leq i \leq d$. □

LEM 59 For $0 \leq i \leq d$

$$(i) \quad \tau_i(A) = \sum_{h=i}^d \tau_i(\theta_h) E_h$$

$$(ii) \quad \eta_i(A) = \sum_{h=0}^{d-i} \eta_i(\theta_h) E_h$$

pf (i)

$$\tau_i(A) = \tau_i(A) I$$

$$= \tau_i(A) \sum_{h=0}^d E_h$$

$$= \sum_{h=0}^d \tau_i(\theta_h) E_h$$

$$\left[\tau_i(\theta_h) = 0 \text{ for } 0 \leq h \leq i-1 \right]$$

$$= \sum_{h=i}^d \tau_i(\theta_h) E_h$$

(ii) Apply (i) to $\Phi \downarrow$

□

LEM 60 For $0 \leq i \leq d$.

(i) $\{\pi_j(A)\}_{j=0}^i$ and $\{A^j\}_{j=0}^i$ have same span

(ii) $\{\pi_j(A)\}_{j=i}^d$ and $\{E_j\}_{j=i}^d$ have same span.

pf (i) Poly π_j has degree j for $0 \leq j \leq d$.

(ii) By LEM 59

□

LEM 61 For $0 \leq i \leq d$

(i) $\{\pi_j(A)\}_{j=0}^i$ and $\{A^j\}_{j=0}^i$ have same span

(ii) $\{\pi_j(A)\}_{j=i}^d$ and $\{E_j\}_{j=i}^d$ have same span.

pf Apply LEM 60 to $\mathbb{F}^{\mathbb{F}}$

□

We continue to discuss a PLS on V :

$$\Phi = (A, \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

DEF 62 We say E_0^* is normalizing whenever

$$E_i E_0^* \neq 0 \quad (0 \leq i \leq d)$$

LEM 63 TFAE:

(i) E_0^* is normalizing

(ii) $\mathcal{D} E_0^*$ has dim $d+1$

(iii) the elements $\{A^i E_0^*\}_{i=0}^d$ are lin indep

(iv) $\forall X \in \mathcal{D},$

$$X E_0^* = 0 \text{ implies } X = 0$$

pf Routine

□

LEM 64 TFAE

(i) E_0^* is normalizing(ii) $E_i V = E_i E_0^* V \quad (0 \leq i \leq d)$ (iii) $\mathcal{O} E_0^* V = V$ (iv) For $0 \neq v \in E_0^* V$ the map

$$\mathcal{O} \rightarrow V$$

$$X \rightarrow Xv$$

is a bijection

pf Routine

□

Prop 65 Assume E_0^* is normalizing. Then
 $\forall X \in \text{End}(V)$ and $0 \leq i \leq d$, IFAE:

$$(i) \quad X E_i = 0$$

$$(ii) \quad X E_i E_0^* = 0$$

$$\begin{aligned} \text{pf} \quad X E_i = 0 &\Leftrightarrow X E_i V = 0 \\ &\quad \underbrace{E_i E_0^* V}_{E_i E_0^* V} \\ &\Leftrightarrow X E_i E_0^* V = 0 \\ &\Leftrightarrow X E_i E_0^* = 0 \end{aligned}$$

□

DEF 66 By a decomposition of V we mean a
 sequence $\{V_i\}_{i=0}^d$ of 1-dim'l subspaces s.t.

$$V = \sum_{i=0}^d V_i \quad (\text{direct sum})$$

Obs Given a basis $\{v_i\}_{i=0}^d$ of V

Let $V_i = \text{Span}(v_i)$ $0 \leq i \leq d$.

Then $\{V_i\}_{i=0}^d$ is a decomp of V

We say $\{V_i\}_{i=0}^d$ is induced by $\{v_i\}_{i=0}^d$.

LEM 6.1 Assume E_0^* is normalizing, and
define

$$U_i = \tau_i(A) E_0^* V \quad (0 \leq i \leq d)$$

Then

(i) $\{U_i\}_{i=0}^d$ is a decomp of V

(ii) $(A - \theta_i I) U_i = U_{i+1} \quad (0 \leq i \leq d-1)$

(iii) $(A - \theta_d I) U_d = 0$

(iv) $U_0 + U_1 + \dots + U_i = E_0^* V + A E_0^* V + \dots + A^i E_0^* V \quad (0 \leq i \leq d)$

(v) $U_i + U_{i+1} + \dots + U_d = E_i V + E_{i+1} V + \dots + E_d V \quad (0 \leq i \leq d)$

pf (i) $\{T_i(A)\}_{i=0}^d$ basis for $\mathcal{D}$

For $0 \neq v \in E_0^* V$

$\{T_i(A)v\}_{i=0}^d$ basis for $\mathcal{D} E_0^* V = V$

so

$\{U_i\}_{i=0}^d$ decomp of V

(ii) Since

$$(\lambda - \theta_i) T_i(\lambda) = T_{in}(\lambda) \quad (0 \leq i \leq d-1)$$

$$(iii) \quad (\lambda - \theta_d) T_d(\lambda) = (\lambda - \theta_0)(\lambda - \theta_1) \dots (\lambda - \theta_{d-1}) \\ = \text{min poly of } A$$

$$so \quad (A - \theta_d I) T_d(A) = 0$$

$$so \quad 0 = (A - \theta_d I) \underbrace{T_d(A) E_0^* V}_{U_d}$$

(iv) Since $T_i(\lambda)$ has degree i $0 \leq i \leq d$

(v) Since $\text{span}\{T_1(A), \dots, T_d(A)\}$

$$= \text{span}\{E_1, \dots, E_d\}$$

for $0 \leq i \leq d$

□

LEM 68 TFAE

(i) $\forall 0 \leq i, j \leq d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases}$$

(ii) $\forall 0 \leq i \leq d \exists f_i \in \mathbb{F}[\lambda]$ with $\deg f_i = i$

and

$$E_i^* V = f_i(A) E_0^* V$$

(iii) for $0 \leq i \leq d$

$$E_0^* V + \dots + E_i^* V = E_0^* V + A E_0^* V + \dots + A^i E_0^* V$$

Assume (i) - (iii) hold. Then E_0^* is normalizing.pf (i) $\rightarrow$ (iii) $\forall 0 \leq i \leq d$ pick

$$0 \neq v_i \in E_i^* V$$

So

 $\{v_i\}_{i=0}^d$ is basis for V

let $B \in \text{Mat}_d(\mathbb{F})$ represent A wrt $\{v_i\}_{i=0}^d$

$$B = \begin{pmatrix} & & * \\ & & \\ 0 & & \end{pmatrix}$$

↑ subdiagonal entries
non 0

$$Av_0 = -v_0 + \frac{1}{\#} v_1$$

$$v_1 = f_1(A) \quad f_1 \in \mathbb{F}[\lambda] \quad \deg f_1 = 1$$

$$Av_1 = -v_0 + -v_1 + \frac{1}{\#} v_2$$

$$v_2 = f_2(A) \quad f_2 \in \mathbb{F}[\lambda] \quad \deg f_2 = 2$$

etc
 B is indecomposable

$$v_i = f_i(A)v_0 \quad f_i \in \mathbb{F}[\lambda] \quad \deg f_i = i$$

$$\text{So } E_i^* v = f_i(A) E_0^* v \quad 0 \leq i \leq d$$