

Math 846

Lecture 2

D=3 guess

eigenvalue	3	1	-1	-3
mult	1	3	3	1

Prop 5 For the hypercube $\Gamma = H(n, 2)$

The eigenvalues / mults are

$$\theta_i = D - 2i \quad (0 \leq i \leq D),$$

$$m_i = \binom{D}{i} \quad (0 \leq i \leq D).$$

pf View $X = X_0$ as Cartesian product

$$X_0 = X_1 \times X_1 \times \dots \times X_1 \quad (D \text{ copies})$$

$$X_1 = \{1, -1\}$$

View $V = V_0$ as tensor product

$$V_0 = V_1 \otimes V_1 \otimes \dots \otimes V_1 \quad (D \text{ copies})$$

$$\otimes = \otimes_{\mathbb{F}}$$

$$V_1 = \text{span}\{\hat{i}, \hat{j}\}$$

$A = A_0$ acts on V_D as follows,

$$\text{For } u = u_1 \otimes u_2 \otimes \dots \otimes u_D \in V_D,$$

$$A_D(u_1 \otimes u_2 \otimes \dots \otimes u_D) = \sum_{i=1}^D u_1 \otimes \dots \otimes u_{i-1} \otimes A_i u_i \otimes u_{i+1} \otimes \dots \otimes u_D$$

where

$$A_i \hat{1} = \hat{1}$$

$$A_i \hat{(-1)} = \hat{(-1)}$$

Define vectors in V_i :

$$1^V = \hat{1} + \hat{(-1)}$$

$$(-1)^V = \hat{1} - \hat{(-1)}$$

obs

$$A_i 1^V = 1^V$$

$$A_i (-1)^V = -(-1)^V$$

vectors $1^V, (-1)^V$ form basis for V_i

V_D has basis

$$u_1 \otimes u_2 \otimes \dots \otimes u_D$$

$$u_i \in \{1^V, (-1)^V\} \quad (1 \leq i \leq D) \quad (\star)$$

consisting of eigenvectors for A_D .

For a vector $u = u_1 \otimes u_2 \otimes \dots \otimes u_D$ in basis $\star$

write

$$\underset{\substack{\parallel \\ l}}{\text{index}(u)} = \left| \left\{ i \mid 1 \leq i \leq D, u_i = (-1)^{v_i} \right\} \right|$$

By construction,

$$A_D u = \theta u$$

where

$$\begin{aligned} \theta &= \left| \left\{ i \mid 1 \leq i \leq D, u_i = 1^{v_i} \right\} \right| \\ &\quad - \left| \left\{ i \mid 1 \leq i \leq D, u_i = (-1)^{v_i} \right\} \right| \\ &= (D - l) - l \\ &= D - 2l \end{aligned}$$

For $0 \leq l \leq D$

basis elements in $\star$ with index l

$$= \# \text{ } l\text{-subsets of } \{1, 2, \dots, D\}$$

$$= \binom{D}{l}$$

So eigenvalue $D - 2l$ has mult $\binom{D}{l}$.

The corresp eigenspace has a basis consisting of the vectors in $\star$ that have index l



Any graph $\Gamma = (X, E)$ is called bipartite whenever

$\exists$ bipartition

$$X = X^+ \cup X^- \quad (\text{disjoint union})$$

st adjacent vertices are in opposite parts of the bipartition.

The graph $\Gamma = H(0, 2)$ is bipartite, with

$$X^+ = \{ (a_1, a_2, \dots, a_d) \in X \mid a_1, a_2, \dots, a_d > 0 \}$$

$$X^- = \{ (a_1, a_2, \dots, a_d) \in X \mid a_1, a_2, \dots, a_d < 0 \}$$

Next goal: for $\Gamma = H(0, 2)$ find $\text{Aut}(\Gamma)$

ex $D=1$ $\Gamma:$ 
 $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ with 2 elements
 $D=2$ $\Gamma:$ 
 $\text{Aut}(\Gamma)$ is dihedral group with 8 elements
General $D \geq 1$:

A signed permutation matrix of order D

is a $D \times D$ matrix P such that

(i) each entry of P is in $\{0, 1, -1\}$

(ii) each row of P has exactly one non-zero entry

(iii) each column of P has exactly one non-zero entry

Let $G = G_D$ denote the set of all signed permutation matrices of order D .

G becomes a group under matrix mult.

ex

D	G	G
1	(± 1)	2
2	$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$	8

For general D

$$|G| = 2^D D!$$

G has subgroups

$$H = \left\{ g \in G \mid g \text{ diagonal} \right\}$$

$$= \left\{ \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \right\}$$

so

$$H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2 \quad (D \text{ copies})$$

" $\mathbb{Z}_2^D$

$$S_D = \left\{ g \in G \mid g \text{ has all entries } 1 \right\}$$

"permutation matrices"

= symmetric group of order D

$$|H| = 2^D \quad |S_D| = D!$$

One checks:

G is generated by H, S_D

$H \trianglelefteq G$ (H is normal in G)

$$H \cap S_D = \{\text{id}\}$$

We write

$$G = H \rtimes S_D \quad \text{"semi direct product"}$$

G is called the hyperoctahedral group of order D .

Prop 6 With above notation, $\Gamma = H(2)$

$\text{Aut}(\Gamma)$ is isomorphic to $G = G_D$

pf For notational convenience, view vertices of Γ as column vectors:

$$X = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_0 \end{pmatrix} \mid a_i \in \{1, -1\} \quad 1 \leq i \leq D \right\}$$

G acts on X by left mult

$$G \times X \rightarrow X$$

This induces an (injective) group hom

$$G \rightarrow \text{Aut}(\Gamma)$$

For notational convenience, identify G with its image via this hom.

Show $G = \text{Aut}(\Gamma)$.

Fix $x \in X$, say $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

obs

$$X = \{ h_x \mid h \in H \}$$

" H is trans on X "

obs

$$S_D = \{ g \in G \mid g \cdot x = x \} \quad \text{"stabilizer of } x \text{ in } G"$$

Write

$$\Gamma(x) = \{ x_1, x_2, \dots, x_n \}$$

where

$$x_i = \begin{pmatrix} 1 \\ \vdots \\ i \\ \vdots \\ n \end{pmatrix} \leftarrow i$$

Group S_D acts faithfully on $\Gamma(x)$ as the group of all permutations of $\Gamma(x)$.

Assume $\exists \sigma \in \text{Aut}(\Gamma) \setminus G$

We will get a contradiction.

Mult σ by element of H , WLOG

$$\sigma \cdot x = x$$

Mult σ by element of S_D , WLOG

$$\sigma \cdot x = x \text{ and } \sigma y = y \quad \forall y \in \Gamma(x)$$

For $z \in X$ show $\sigma z = z$

write

$$z = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_0 \end{pmatrix}$$

For $1 \leq i \leq p$,

$$\partial(x_i, z) = \begin{cases} \partial(x_i, z) + 1 & \text{if } a_i = 1 \\ \partial(x_i, z) - 1 & \text{if } a_i = -1 \end{cases}$$

So
$$a_i = \partial(x_i, z) - \partial(x_i, z)$$

We can recover z from data

$$\partial(x_i, z) - \partial(x_i, z) \quad (1 \leq i \leq p)$$

obs for $1 \leq i \leq p$

$$\begin{aligned} & \partial(x_i, \sigma z) - \partial(x_i, \sigma z) \\ &= \partial(\sigma(x_i), \sigma(z)) - \partial(\sigma(x_i), \sigma(z)) \\ &= \partial(x_i, z) - \partial(x_i, z) \end{aligned}$$

So $\sigma z = z$ cont.

Therefore

$$G = \text{Aut}(P)$$

□

The graph $\Gamma = H(0, 2)$ is transitive, since

$H \leq \text{Aut}(\Gamma)$ is trans on X

Next goal: show Γ is distance-transitive.

LEM 7 For the vertex $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ in $\Gamma = H(0, 2)$

the subgroup $S_0 \leq \text{Aut}(\Gamma)$ is transitive on $\Gamma_i(x)$

for $0 \leq i \leq 0$.

pf the set

$$\Gamma_i(x) = \{y \in X \mid d(x, y) = i\}$$

is invariant under $S_0 = \text{Stab}(x)$

Pick $y \in \Gamma_i(x)$, say

$$y = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \in \Gamma_i$$

Pick $y' \in \Gamma_i(x)$ and write

$$y' = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_0 \end{pmatrix}$$

exactly i of the $a_1, a_2, \dots, a_0$ are -1

Elements of S_0 permute coordinates

$\exists \sigma \in S_0$ st

$$\sigma(y) = y'$$

□

Prop 8 The graph $\Gamma = H(0, 2)$ is distance-transitive

pf Pick $x, y \in X$ and write $i = \partial(x, y)$.

Define

$$x' = \begin{pmatrix} 1 \\ \vdots \\ i \\ \vdots \\ 1 \end{pmatrix} \quad y' = \begin{pmatrix} -1 \\ \vdots \\ -i \\ \vdots \\ -1 \end{pmatrix} \} i$$

$$\text{so } \partial(x', y') = i$$

Display $\sigma \in \text{Aut}(\Gamma)$ st $\sigma(x) = x'$ and $\sigma(y) = y'$.

Since $H \subseteq \text{Aut}(\Gamma)$ is trans on X , $\exists \sigma_1 \in H$ st $\sigma_1(x) = x'$.

$$\text{Obs } i = \partial(x, y) = \partial(\sigma_1(x), \sigma_1(y)) = \partial(x', \sigma_1(y))$$

By Lem 7, $\exists \sigma_2 \in S_0$ that sends

$$\sigma_1(y) \rightarrow y'$$

Now $\sigma = \sigma_2 \sigma_1$ sends

$$\begin{array}{ccccc} x & \xrightarrow{\sigma_1} & x' & \xrightarrow{\sigma_2} & x' \\ & & \sigma_1 & & \sigma_2 \end{array} \quad \begin{array}{ccccc} y & \xrightarrow{\sigma_1} & \sigma_1(y) & \xrightarrow{\sigma_2} & y' \\ & & \sigma_1 & & \sigma_2 \end{array}$$

□