

Math 846

Lecture 19

Recall:

- $\mathbb{F}$  is arb
- Fix  $d \in \mathbb{N}$
- vector space  $V$  has  $\dim d+1$

We saw, a Leonard pair on  $V$  is the same thing as a tridiagonal pair on  $V$  that has all eigenspaces of  $\dim 1$ .

By a Leonard system on  $V$  we mean a tridiagonal system on  $V$  that has shape

$$p_i = 1 \quad 0 \leq i \leq d$$

Here is another point of view.

Ex 45 A Leonard system on  $V$  is a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

of elements in  $\text{End}(V)$  st:

(LS1) each of  $A, A^*$  is mult free

(LS2)  $\{E_i\}_{i=0}^d$  is an ordering of the prim idempotents of  $A$

(LS3)  $\{E_i^*\}_{i=0}^d$  is an ordering of the prim idempotents of  $A^*$

(LS4)  $\forall 0 \leq i, j \leq d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$$

(LS5)  $\forall 0 \leq i, j \leq d$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$$

(1) DEF 46 Let  $A, A^*$  denote a Leonard pair (LP) on  $V$ , and let  $B, B^*$  denote a LP on a vector space  $V'$ .

By an isomorphism of Leonard pairs from  $A, A^*$  to  $B, B^*$

we mean a vector space iso

$$\sigma: V \rightarrow V'$$

st

$$B = \sigma A \sigma^{-1},$$

$$B^* = \sigma A^* \sigma^{-1}$$

the LP  $A, A^*$  and  $B, B^*$  are called isomorphic

whenever  $\exists$  iso of LP from  $A, A^*$  to  $B, B^*$

Notation let  $\sigma: V \rightarrow V'$  denote an iso of vector spaces.

For  $X \in \text{End}(V)$  write

$$X^\sigma = \sigma X \sigma^{-1}$$

Note

$$X^\sigma \in \text{End}(V')$$

The map

$$\text{End}(V) \rightarrow \text{End}(V')$$

$$X \rightarrow X^\sigma$$

is iso of algebras.

For a Leonard system (LS)

$$\Phi = (A, \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

on  $V$  the sequence

$$\Phi^\sigma = (A^\sigma, \{E_i^\sigma\}_{i=0}^d; A^{*\sigma}, \{E_i^{*\sigma}\}_{i=0}^d)$$

is LS on  $V'$

DEF 47 let  $\Phi$  denote a LS on  $V$ . let  $\Phi'$  denote a LS on a vector space  $V'$ .

By an iso of LS from  $\Phi$  to  $\Phi'$  we mean a vector space iso  $\sigma: V \rightarrow V'$  s.t.

$$\Phi^\sigma = \Phi'$$

The LS  $\Phi$  and  $\Phi'$  are called isomorphic whenever

$\exists$  iso of LS from  $\Phi$  to  $\Phi'$ .

DEF 48 By a Pre Leonard system on  $V$

we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

of elements in  $\text{End}(V)$  that satisfy LS1, LS2, LS3

DEF 49 For the above PLS  $\Phi$

• for  $0 \leq i \leq d$  let  $\theta_i =$  eigenvalue of  $A$  for  $E_i$   
 $\theta_i^* = \dots$   $A^*$  for  $E_i^*$

• call  $\{\theta_i\}_{i=0}^d$  the eigenvalue sequence of  $\Phi$

• call  $\{\theta_i^*\}_{i=0}^d$  the dual eigenvalue sequence of  $\Phi$

• let  $\mathcal{D} =$  subalg of  $\text{End}(V)$  gen by  $A$

(so  $\mathcal{D}$  has basis  $\{A^i\}_{i=0}^d$  and basis  $\{E_i\}_{i=0}^d$ )

• let  $\mathcal{D}^* =$  subalg of  $\text{End}(V)$  gen by  $A^*$

DEF 50 For the above PLS  $\Phi$ , define

$$a_i = \text{tr}(A E_i^X)$$

$$a_i^Y = \text{tr}(A^Y E_i^Y)$$

$$0 \leq i \leq d$$

- call  $\{a_i\}_{i=0}^d$  the diagonal sequence of  $\Phi$
- call  $\{a_i^Y\}_{i=0}^d$  the dual diagonal sequence of  $\Phi$

LEM 51 For the above PLS  $\Phi$

$$\sum_{i=0}^d \theta_i = \sum_{i=0}^d a_i$$

$$\sum_{i=0}^d \theta_i^Y = \sum_{i=0}^d a_i^Y$$

pf

$$\sum_{i=0}^d \theta_i = \text{tr}(A)$$

$$= \text{tr}(A \cdot I)$$

$$= \text{tr}\left(A \sum_{i=0}^d E_i^X\right)$$

$$= \sum_{i=0}^d a_i$$

□

LEM 52 For the above PLS  $\Phi$ ,

$$(i) \quad E_i^* A E_i^* = \alpha_i E_i^* \quad (0 \leq i \leq d)$$

$$(ii) \quad E_i A^* E_i = \alpha_i^* E_i \quad (0 \leq i \leq d)$$

pf (i) Abbr  $A = \text{End}(V)$

Since  $E_i^*$  has rank 1,

$$E_i^* A E_i^* = \text{Span}(E_i^*)$$

So  $\exists \alpha_i \in \mathbb{F}$  st

$$E_i^* A E_i^* = \alpha_i E_i^*$$

Take trace

$$\text{tr}(E_i^* A E_i^*) = \alpha_i \underbrace{\text{tr}(E_i^*)}_{=1} = \alpha_i$$

$$\text{tr}(A E_i^* E_i^*)$$

$$\text{tr}(A E_i^*)$$

$$\text{tr}$$

$$\alpha_i$$

□

For the above PLS

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

each of the following is a PLS on  $V$ :

$$\Phi^* := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$$

$$\Phi^\downarrow := (A; \{E_i\}_{i=0}^d; A^*, \{E_{d-i}^*\}_{i=0}^d)$$

$$\Phi^\Downarrow := (A, \{E_{d-i}\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

|                   | eigenvalue<br>sequence | dual eigen<br>sequence | diagonal<br>sequence | dual diag<br>sequence |
|-------------------|------------------------|------------------------|----------------------|-----------------------|
| $\Phi$            | $\theta_i$             | $\theta_i^*$           | $a_i$                | $a_i^*$               |
| $\Phi^*$          | $\theta_i^*$           | $\theta_i$             | $a_i^*$              | $a_i$                 |
| $\Phi^\downarrow$ | $\theta_i$             | $\theta_{d-i}^*$       | $a_{d-i}$            | $a_i^*$               |
| $\Phi^\Downarrow$ | $\theta_{d-i}$         | $\theta_i^*$           | $a_i$                | $a_{d-i}^*$           |

LEM 53 For the above PLS  $\Phi$ , assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$$



Then the elements

$$A^i E_0^* A^j \quad 0 \leq i, j \leq d$$

form a basis for the vector space  $\text{End}(V)$ .

pf For  $0 \leq i \leq d$  pick  $0 \neq v_i \in E_i^* V$ .

So  $\{v_i\}_{i=0}^d$  is a basis for  $V$ .

WLOG we identify each  $X \in \text{End}(V)$  with

the matrix in  $\text{Mat}_{\text{dim}(V)}(\mathbb{F})$  that represents  $X$  wrt  $\{v_i\}_{i=0}^d$

Then  $A$  becomes irreducible tridiagonal matrix and

$$E_0^* \text{ becomes } \text{diag}(1, 0, \dots, 0)$$

For  $0 \leq i, j \leq d$  and  $0 \leq r, s \leq d$

$$(A^i E_0^* A^j)_{rs} = (A^i)_{r0} (A^j)_{0s}$$

We have

$$(A^i)_{rc} = \begin{cases} 0 & \text{if } r > c \\ \neq 0 & \text{if } r = c \end{cases}$$

$$(A^j)_{sa} = \begin{cases} 0 & \text{if } s > a \\ \neq 0 & \text{if } s = a \end{cases}$$

$$A^i E_0^* A^j = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \quad \bullet \neq 0$$

So

$$A^i E_0^* A^j \quad 0 \leq i, j \leq d$$

are linearly indep.

Result follows since  $\dim(\text{End}(V)) = (d+1)^2$

□

LEM 54 For the above PLS  $\Phi$ , assume  $\star$ . Then

(i)  $A, E_0^*$  is a generating set for the algebra  $\text{End}(V)$

(ii)  $A, A^*$  " " "

pf (i) By Lem 53

(ii) By (i) and since  $E_0^*$  is a polynomial in  $A^*$ .  $\square$

### Notation

For any algebra  $A$

• By an automorphism of  $A$  we mean an algebra isomorphism  $A \rightarrow A$ .

• By an antiautomorphism of  $A$  we mean a vector space iso  $\sigma: A \rightarrow A$  st

$$\sigma(xy) = \sigma(y)\sigma(x) \quad \forall x, y \in A$$

LEM55 For the above PLS  $\Phi$ , assume  $\star$ .

Then

(i)  $\exists$  unique antiautomorphism  $\dagger$  of  $\text{End}(V)$  that fixes each of  $A, A^*$

(ii)  $\dagger$  fixes everything in  $\mathcal{D}$  and everything in  $\mathcal{D}^*$

(iii)  $\dagger$  fixes each of  $E_i, E_i^*$  for  $0 \leq i \leq d$

(iv)  $\dagger^2 = \text{id}$

pf (i) First we show  $\dagger$  exists

For  $0 \leq i \leq d$  pick  $0 \neq v_i \in E_i^* V$

So  $\{v_i\}_{i=0}^d$  is a basis for  $V$

For  $X \in \text{End}(V)$  let  $X^\# \in \text{Mat}_{d+1}(\mathbb{F})$  represent

$X$  wrt  $\{v_i\}_{i=0}^d$ .

The map

$$\begin{aligned} \# : \quad \text{End}(V) &\rightarrow \text{Mat}_n(\mathbb{F}) \\ X &\rightarrow X^\# \end{aligned}$$

is algebra is o.

Write  $B = A^\# \quad B^* = (A^*)^\#$

The matrix  $B$  is irred tridiagonal and

$$B^t = \text{diag}(\theta_1^*, \dots, \theta_d^*)$$

Define a diagonal matrix  $K \in \text{Mat}_n(\mathbb{F})$

with  $(i,i)$ -entry

$$K_{ii} = \frac{B_{0i} B_{i2} \cdots B_{i,i+d}}{B_{i0} B_{i1} \cdots B_{i,i+d}} \quad (0 \leq i \leq d)$$

then  $K$  is invertible and

$$K^* B^t K = B$$

Define a map

$$b : \text{Mat}_n(\mathbb{F}) \rightarrow \text{Mat}_n(\mathbb{F})$$

$$X \rightarrow K^{-1} X^t K$$

Obs  $b$  is an anti-autom of  $\text{Mat}_n(\mathbb{F})$   
 that fixes each of  $B, B^*$ .

The composition

$$t : \text{End}(V) \xrightarrow{\#} \text{Mat}_n(\mathbb{F}) \xrightarrow{b} \text{Mat}_n(\mathbb{F}) \xrightarrow{\#^{-1}} \text{End}(V)$$

is an anti-autom of  $\text{End}(V)$  that fixes each of  $A, A^*$ .

(ii) By (i)

(iii) For  $0 \leq i \leq d$

$$E_i \in \mathcal{D}$$

$$E_i^* \in \mathcal{D}^*$$

(iv)  $t^2$  is an automorphism of  $\text{End}(V)$  that fixes  $A, A^*$   
 But  $A, A^*$  generate  $\text{End}(V)$  so  $t^2 = \text{id}$ . □