

Math 846

Lecture 18

THM 44 Assume $\mathbb{F}$ is urbo.

Given $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$

Assume q is not a root of unity where $\beta = q + q^*$

Let V denote a finite dim'l irred module for

$$T = T(\beta, \gamma, \gamma^*, \delta, \delta^*),$$

Assume that the algebra generators A, A^* are diagonalizable on V .

Then A, A^* act on V as a tridiagonal pair.

pf Let Ω denote the set of eigenvalues for A on V .

For $\theta \in \Omega$ let V_θ denote the corresp eigenspace of A and E_θ the corresp prim idempotent.

Define polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

Given distinct $\theta, \tau \in \Omega$ st

$$E_\theta A^* E_\tau \neq 0$$

we show

$$P(\theta, \tau) = 0$$

obs

$$0 = E_\theta \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right] E_\tau$$

$$= E_\theta A^* E_\tau (\theta - \tau) (\theta^2 - \beta \theta \tau + \tau^2 - \gamma(\theta + \tau) - \delta)$$

$$= \underbrace{E_\theta A^* E_\tau}_{\neq 0} \underbrace{(\theta - \tau)}_{\neq 0} \underbrace{P(\theta, \tau)}_{\text{must be 0}}$$

For $\theta, \tau \in \Omega$ define θ, τ to be adjacent

whenever

$$\theta \neq \tau \text{ and } P(\theta, \tau) = 0$$

Adjacency is a symmetric binary relation on Ω which turns Ω into a graph

this graph is connected since the T-module V is used.

Since the polynomial P is quadratic in each argument,

each element of Ω is adjacent at most 2 elements of Ω

So the graph Ω is a path or a cycle.

We show Ω is a path.

Suppose not. Then Ω is a cycle with $|\Omega| \geq 3$

Write $d = |\Omega| - 1$

Order the elements in Ω

$$\theta_0, \theta_1, \dots, \theta_d$$

st

θ_{i-1}, θ_i are adjacent ($1 \leq i \leq d$) and θ_d, θ_0 adj.

Define $\theta_{d+1} = \theta_0$

The sequence

$$\theta_0, \theta_1, \dots, \theta_{d+1}$$

satisfies

$$P(\theta_i, b_i) = 0 \quad 1 \leq i \leq d+1$$

So

$$\theta_0, \theta_1, \dots, \theta_{d+1}$$

is (β, γ, δ) -rec and hence β -rec.

Note $\beta \neq \mathbb{F}_2$ since $q \neq \mathbb{F}_1$

So $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^{\text{cl}}$ st

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \quad (0 \leq i \leq d+1)$$

obs

$$0 = \theta_{d+1} - \theta_0$$

$$= \alpha_2 (q^{d+1} - 1) + \alpha_3 (q^{-d-1} - 1)$$

$$= \underbrace{(q^{d+1} - 1)}_{\neq 0} (\alpha_2 - \alpha_3 q^{-d-1})$$

So

$$\alpha_3 = \alpha_2 q^{d+1}$$

Now

$$\begin{aligned}
 \theta_d - a &= \alpha_1 + \alpha_2 q^d + \alpha_3 q^{-d} \\
 &\quad - (\alpha_1 + \alpha_2 q + \alpha_3 q^{-1}) \\
 &= \alpha_2 (q^d - q) + \alpha_3 (q^{-d} - q^{-1}) \\
 &\quad \parallel \\
 &\quad \alpha_2 q^{d+1} \\
 &= 0
 \end{aligned}$$

emb.

The graph Ω is a path.

Write

$$V_i = V_{\partial_i} \quad E_i = E_{\partial_i} \quad (0 \leq i \leq d)$$

By ansatz

$$E_i A^* E_j = 0 \quad \text{if } |i-j| > 1 \quad (0 \leq i, j \leq d)$$

So

$$A^* V_i \leq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d)$$

where $V_{-1} = 0, \quad V_{d+1} = 0$

Exchange the roles of A, A^* :

$\exists$ ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^*

st

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta)$$

where

$$V_{-1}^* = 0 \quad \text{and} \quad V_{\delta+1}^* = 0,$$

By these comments A, A^* act on V as a

PD pair.

□

Comments about TD pairs

$\mathbb{F}$ arb $0 \neq V = \text{fd vs } \mathbb{F}$

Given $\lambda, A^* \in \text{End}(V)$ that are diagonalizable

and $\exists$ subspace $W \subseteq V$ st

$$W \neq 0, W \neq V, A W \subseteq W, A^* W \subseteq W$$

☆

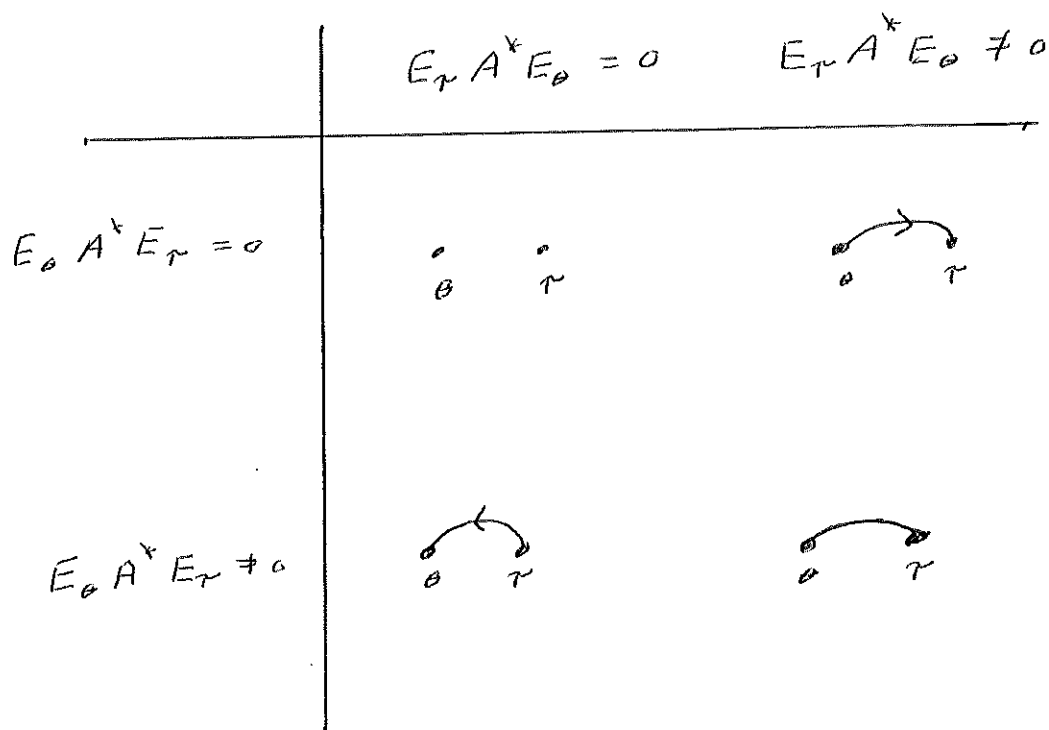
Using A, A^* we create two diagrams Ω and Ω^* .

let $\Omega = \text{set of eigenvalues of } A$

For $\theta \in \Omega$ let $V_\theta = \theta$ -eigenspace of A

and $E_\theta = \theta$ -primitive idempotent for A

For distinct $\theta, \tau \in \Omega$ we draw



View Ω as "directed" graph.

By $\star$, $\forall$ dist $\theta, \tau \in \Omega$

$\exists$ "directed" walk from θ to τ :

A sequence $\{\theta_i\}_{i=0}^r$ $\theta_i \in \Omega$

st

$$\theta_0 = \theta, \quad \theta_r = \tau$$

$$\theta_{i-1} \rightarrow \theta_i \quad \text{or} \quad \theta_{i-1} \dashrightarrow \theta_i \quad (1 \leq i \leq r)$$

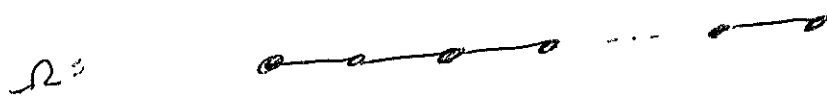
Call Ω the (A, A^*) -diagram.

Let Ω^* denote the (A^*, A) -diagram.

Obs TFAE

(i) A, A^* is a TD pair on V

(ii) each of Ω, Ω^* is a path:



Moreover if (i), (ii) holds then

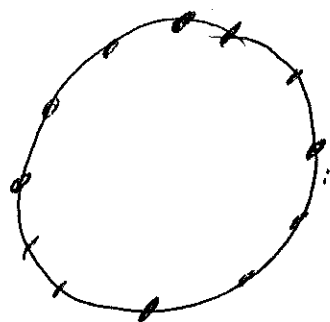
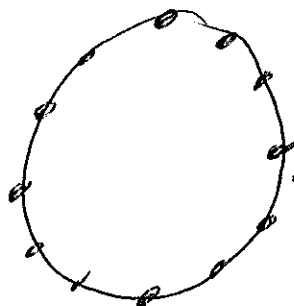
$$|\Omega| = |\Omega^*|.$$

Open problems (research level)

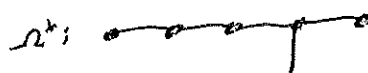
- What pairs of diagrams Ω, Ω^* are possible?

[for example, two paths of different lengths are impossible]

- study/classify the pairs A, A^* such that each of Ω, Ω^* is a cycle;

 Ω  Ω^*

- study/classify the pairs A, A^* such that each of Ω, Ω^* is a tree;

 Ω  Ω^*

- Let P = any property that a square matrix might have

Define a P -pair on V to be an ordered pair $A, A^* \in \text{End}(V)$ st

- (i) $\exists$ basis for V wrt which the matrix rep A is diagonal and the matrix rep A^* has P
- (ii) $\exists$ basis for V wrt which the matrix rep A has P and matrix rep A^* is diagonal.

study/classify the P -pairs for your favorite property P .

Next goal: classify the Leonard pairs

Until further notice:

- Field $\mathbb{F}$ is arbitrary
- every vector space and algebra discussed is understood to be over $\mathbb{F}$.
- We fix $d \in \mathbb{N}$.
- $\text{Mat}_d(\mathbb{F})$ denotes the (associative) algebra of $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{F}$.
We index the rows and columns by $0, 1, 2, \dots, d$
- V denotes a vector space with $\dim d+1$.

We recall how each basis $\{v_i\}_{i=0}^d$ of V

gives an algebra iso $\text{End}(V) \rightarrow \text{Mat}_d(\mathbb{F})$.

For $X \in \text{End}(V)$ and $B \in \text{Mat}_d(F)$, we say

B represents X wrt $\{v_i\}_{i=0}^d$

whenever

$$X v_j = \sum_{i=0}^d B_{ij} v_i \quad (0 \leq j \leq d).$$

The algebra iso sends X to the unique

matrix B in $\text{Mat}_d(F)$ that represents X wrt $\{v_i\}_{i=0}^d$

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