

Math 846

Lecture 17

$\rightarrow$ $\mathbb{F}$ arb

$$0 \neq V = \text{fd vs} / \mathbb{F}$$

TD system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

[finish pf of Th 38 see notes from Wed]

Prop 39 We refer to THM 38

$$(i) \quad \delta = \theta_{i-2}^2 - \beta \theta_{i-2} \theta_i + \theta_i^2 - \gamma (\theta_{i-2} + \theta_i) \quad 1 \leq i \leq d$$

$$(ii) \quad \delta^* = \theta_{i-2}^{*2} - \beta \theta_{i-2}^* \theta_i^* + \theta_i^{*2} - \gamma^* (\theta_{i-2}^* + \theta_i^*) \quad 1 \leq i \leq d$$

$$(iii) \quad \gamma = \theta_{i-2} - \beta \theta_i + \theta_{i+2} \quad 1 \leq i \leq d-1$$

$$(iv) \quad \gamma^* = \theta_{i-2}^* - \beta \theta_i^* + \theta_{i+2}^* \quad 1 \leq i \leq d-1$$

$$(v) \quad \text{Fa} \quad 2 \leq i \leq d-1,$$

$$\frac{\theta_{i-2} - \theta_{i+2}}{\theta_{i-2} - \theta_i} = \beta \gamma = \frac{\theta_{i-2}^* - \theta_{i+2}^*}{\theta_{i-2}^* - \theta_i^*}$$

pf By LEM 34 and our discussion of sequences that satisfy a 3-term recurrence.

□

THM. 40 For a TD system on V

$$\Phi = (A, \{E_i\}; A^v; \{E_i^v\}_{i=0}^d)$$

the scalars

$$\frac{\theta_{i,2} - \theta_{i,n}}{\theta_{i,1} - \theta_i}, \quad \frac{\theta_{i,2}^v - \theta_{i,n}^v}{\theta_{i,1}^v - \theta_i^v}$$

are equal and independent of i for $2 \leq i \leq d-1$

p f By Prop 39.

□

We briefly return our attention to

a DRG $\Gamma = (X, E)$. Let $F = \mathbb{R}$ or $\mathbb{C}$

Assume Γ is Q -polynomial wrt $\{E_i\}_{i=0}^D$

$$D \geq 1$$

Fix $x \in X$ with $T = T(x)$

• Recall for $0 \leq i \leq D$

$$A_i x^\wedge = E_i^* \mathbb{1} \quad (\text{call this } \mathbb{1}_i)$$

$$A_i^* \mathbb{1} = |x| E_i x^\wedge \quad (\text{call this } \mathbb{1}_i^*)$$

• Each of

$$\{\mathbb{1}_i\}_{i=0}^D, \quad \{\mathbb{1}_i^*\}_{i=0}^D$$

is a basis for

$$M_x^\wedge = M^* \mathbb{1}$$

- Define

$$W = M_x^\pi = M^\pi \mathbb{I}$$

- W is a T -module

"the primary T -module"

- W is irreducible (ex)

- W has

endpt 0

dual endpt 0

diameter D

shape

$$p_i = 1 \quad (0 \leq i \leq D)$$

- With respect to basis $\{\mathbb{I}_i\}_{i=0}^D$

$$A : \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & & \ddots & \\ & & & \ddots & \\ 0 & & & & c_D & a_D \end{pmatrix}$$

$$A^\pi : \text{diag}(\theta_0^\pi, \theta_1^\pi, \dots, \theta_D^\pi)$$

- With respect to basis $\{\pi_i^x\}_{i=0}^D$

$$A: \text{diag}(\theta_0, \theta_1, \dots, \theta_D)$$

$$A^x: \begin{pmatrix} a_0^x & b_0^x & & & 0 \\ c_1^x & a_1^x & b_1^x & & \\ & c_2^x & a_2^x & b_2^x & \\ & & \ddots & \ddots & \ddots \\ 0 & & & c_{D-1}^x & a_{D-1}^x & b_{D-1}^x \\ & & & & c_D^x & a_D^x \end{pmatrix}$$

- A, A^x act on W as a Leonard pair

- By Th 40,

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_{i+1} - \theta_i},$$

$$\frac{\theta_{i+2}^x - \theta_{i+1}^x}{\theta_{i+1}^x - \theta_i^x}$$

are equal and indep of i for $2 \leq i \leq D-1$

Thm 41 For a DRG $\Gamma = (X, E)$ that is
 $\mathbb{Q}$ -poly w.r.t $\{E_i\}_{i=0}^D$, Fix $x \in X$ and write $T = T(x)$.
 then adj matrix A and dual adj matrix $A^* = A^*(x)$
 satisfy the TD relations

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*], \quad \text{TD1}$$

$$0 = [A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \delta^* A], \quad \text{TD2}$$

where

$$\beta + 1 = \frac{\theta_{i,2} - \theta_{i,n}}{\theta_{i,1} - \theta_i} = \frac{\theta_{i,2}^* - \theta_{i,n}^*}{\theta_{i,1}^* - \theta_i^*} \quad (2 \leq i \leq D+1)$$

$$\gamma = \theta_{i,1} - \beta \theta_i + \theta_{i,n} \quad (1 \leq i \leq D+1)$$

$$\gamma^* = \theta_{i,1}^* - \beta \theta_i^* + \theta_{i,n}^* \quad (1 \leq i \leq D+1)$$

$$\delta = \theta_{i,1}^2 - \beta \theta_{i,1} \theta_i + \theta_i^2 - \gamma(\theta_{i,1} + \theta_i) \quad (1 \leq i \leq D)$$

$$\delta^* = \theta_{i,1}^{*2} - \beta \theta_{i,1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i,1}^* + \theta_i^*) \quad (1 \leq i \leq D)$$

pf let $C = \text{RHS of TD1}$

show $C = 0$

$$C = \sum_{i=0}^D \sum_{j=0}^D E_i C E_j$$

For $0 \leq i, j \leq D$

$$E_i C E_j = \underbrace{E_i A^* E_j}_{\substack{\parallel \\ 0 \text{ if } |i-j| > 1}} \underbrace{(e_i - e_j)}_{\substack{\parallel \\ 0 \text{ if } i=j}} \underbrace{\left(e_i^2 - \beta e_i e_j + e_j^2 - \gamma(e_i + e_j) - \delta \right)}_{\substack{\parallel \\ 0 \text{ if } |i-j|=1}}$$

$$= 0$$

So $C = 0$ and TD1 holds.

We similarly show TD2 holds. □

We return our attention to general TD pairs and TD systems. $F = \text{arb}$

We mention some special cases of the tridiagonal relations.

• F_n

$$\beta = 2$$

$$\gamma = \gamma^* = 0$$

$$\delta = \delta^* = 4$$

the TD1, TD2 become the Polan / Brady rels

$$[A, [A, [A, A^*]]] = 4[A, A^*]$$

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]$$

• F_n

$$\beta = q^2 + q^{-2}$$

$$\gamma = \gamma^* = 0$$

$$\delta = \delta^* = 0.$$

the TD1, TD2 become the q -Serre relations

$$[A, [A, [A, A^*]_q]_{q^2}] = 0$$

$$[A^*, [A^*, [A^*, A]_q]_{q^2}] = 0.$$

F_m

$$\beta = q^2 + q^{-2}$$

$$\delta = \delta^* = 0$$

$$\delta = \delta^* = -(q^2 - q^{-2})^2$$

The TD1, TD2 become

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A]$$

$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*]$$

" q -Dolan/Grady relations"

DEF 42 For $q \in \mathbb{F}$ st $q^4 \neq 1$, define

the associative algebra $\mathcal{O}_q$ by generators and relations, as follows.

The gens are A, A^* .

The relations are the q Hecke/brady relations.

Call $\mathcal{O}_q$ the q -Onsager algebra

DEF 43 For $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$

the corresp tridiagonal algebra $T = T(\beta, \gamma, \gamma^*, \delta, \delta^*)$

is the assoc alg defined by gens A, A^* subject to

the tridiagonal relations.

We have seen that any TD pair A, A^* on V turns V into an ured module for a tridiagonal algebra $T = T(\beta, \gamma, \gamma^*, \delta, \delta^*)$.

Natural question: Let V denote any finite dim'l ured module for a tridiagonal algebra $T(\beta, \gamma, \gamma^*, \delta, \delta^*)$. Must the algebra generators A, A^* act on V as a TD pair?

The answer is "usually yes" but we need to impose some mild assumptions...

Thm 44 Assume $\mathbb{F}$ is arbitrary

Given $\beta, \delta, \delta^*, \delta, \delta^* \in \mathbb{F}$

Assume q not a root of unity, where $\beta = q + q^q$,

let V denote a f.d. $U_q(\mathfrak{sl}_2)$ module for

$$T = T(\beta, \delta, \delta^*, \delta, \delta^*).$$

Assume that the algebra generators A, A^* are diagonalizable on V ,

then A, A^* act on V as a tridiagonal pair.

pf to come...