

Math 846

Lecture 16

$\mathbb{F}$ arb

$$0 \neq V = \text{fd vs } \mathbb{F}$$

Given TD system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Next general goal:

- show A, A^* satisfy the tridiagonal relations
- show the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and indep of i for $2 \leq i \leq d-1$

Aside about sequences that satisfy
a 3-term recurrence. $\mathbb{F}$ arb

throughout this aside fix an integer $d \geq 1$

let $\{\theta_i\}_{i=0}^d$ denote any sequence of scalars in $\mathbb{F}$.

DEF 39 let $\beta, \gamma, \delta \in \mathbb{F}$

(i) the sequence $\{\theta_i\}_{i=0}^d$ is recurrent whenever

$\theta_{i+1} \neq \theta_i$ ($2 \leq i \leq d-1$) and

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_{i+1} - \theta_i}$$

is indep of i for $2 \leq i \leq d-1$.

(ii) the sequence $\{\theta_i\}_{i=0}^d$ is called β -recurrent

whenever

$$\theta_{i+2} - (\beta + 1)\theta_{i+1} + (\beta + 1)\theta_i - \theta_{i-1} = 0$$

for $2 \leq i \leq d-1$

(iii) The sequence $\{\theta_i\}_{i=0}^d$ is called (β, γ) -recurrent

whenever

$$\theta_{i+1} - \beta \theta_i + \theta_{i-1} = \gamma$$

$$\text{for } 1 \leq i \leq d-1$$

(iv) The sequence $\{\theta_i\}_{i=0}^d$ is called (β, γ, δ) -recurrent

whenever

$$\theta_{i+1}^2 - \beta \theta_{i+1} \theta_i + \theta_i^2 - \gamma(\theta_{i+1} + \theta_i) = \delta$$

$$\text{for } 1 \leq i \leq d.$$

The above concepts are related as follows

TF/E

• $\{\theta_i\}_{i=0}^d$ is rec

• $\theta_{i+1} \neq \theta_i$ ($1 \leq i \leq d-1$) and $\exists \beta \in \mathbb{F}$ st

$\{\theta_i\}_{i=0}^d$ is β -rec

For $\beta \in \mathbb{F}$ TFAE:

- $\{\theta_i\}_{i=0}^d$ is β -rec
- $\exists \gamma \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ is (β, γ) -rec.

Given $\beta, \gamma \in \mathbb{F}$

- Assume $\{\theta_i\}_{i=0}^d$ is (β, γ) -rec. Then $\exists \delta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^d$ is (β, γ, δ) -rec.
- Assume $\{\theta_i\}_{i=0}^d$ is (β, γ, δ) -rec and $\theta_{i-1} \neq \theta_i$ for $1 \leq i \leq d$. Then $\{\theta_i\}_{i=0}^d$ is (β, γ) -rec

Next we give some closed forms.

Assume $\{\theta_i\}_{i=0}^d$ is β -rec

alg closure of $\mathbb{F}$
 $\downarrow$

Case $\beta \neq 2, \beta \neq -2$ Pick $0 \neq q \in \mathbb{F}^{cl}$ st $q + q^{-1} = \beta$

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^{cl}$ st

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \quad (0 \leq i \leq d)$$

Case $\beta=2$ and $\text{char } \mathbb{F} \neq 2$

$$\exists d_1, d_2, d_3 \in \mathbb{F} \text{ st}$$

$$\theta_i = d_1 + d_2 i + d_3 i^2 \quad (0 \leq i \leq d)$$

Case $\beta=-2$ and $\text{char } \mathbb{F} \neq 2$

$$\exists d_1, d_2, d_3 \in \mathbb{F} \text{ st}$$

$$\theta_i = d_1 + d_2 (-1)^i + d_3 i (-1)^i \quad (0 \leq i \leq d)$$

Case $\beta=2$ and $\text{char}(\mathbb{F})=2$

$$\exists d_1, d_2, d_3 \in \mathbb{F} \text{ st}$$

$$\theta_i = d_1 + d_2 i + d_3 \begin{pmatrix} i \\ 2 \end{pmatrix} \quad (0 \leq i \leq d)$$

where we interpret

$$\begin{pmatrix} i \\ 2 \end{pmatrix} = \begin{cases} 1 & \text{if } i \equiv 0, 1 \pmod{4} \\ 4 & \text{if } i \equiv 2, 3 \pmod{4} \end{cases}$$

We are done with the aside.

We return to our PD system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

LEM 35 For $0 \leq i \leq d-1$,

$$\begin{aligned} E_i A^* E_{i+1} - E_{i+1} A^* E_i &= (E_0 + E_1 + \dots + E_i) A^* \\ &\quad - A^* (E_0 + E_1 + \dots + E_i) \end{aligned}$$

pf For $0 \leq j \leq d-1$

$$\begin{aligned} E_j A^* &= E_j A^* (E_0 + \dots + E_d) \\ &= E_j A^* (E_{i+1} + E_i + E_{i+1}) \end{aligned} \quad (E_d = 0)$$

Similarly

$$A^* E_j = (E_{i+1} + E_i + E_{i+1}) A^* E_j$$

Sum the above over $j = 0, 1, \dots, i$ and take the difference between the two sums. □

LEM 3b Let M denote the subalgebra
of $\text{End}(V)$ generated by A . Then

$$\begin{aligned} \text{Span} \{ XA^*Y - YA^*X \mid X, Y \in M \} \\ = \{ ZA^* - A^*Z \mid Z \in M \} \end{aligned}$$

pf Recall $\{E_i\}_{i=0}^d$ is a basis for M .

Obs $\{E_0 + \dots + E_i\}_{i=0}^d$ is a basis for M

Obs

$$\begin{aligned} & \text{Span} \{ XA^*Y - YA^*X \mid X, Y \in M \} \\ &= \text{Span} \{ E_i A^* E_j - E_j A^* E_i \mid 0 \leq i, j \leq d \} \\ &= \text{Span} \{ E_i A^* E_m - E_m A^* E_i \mid 0 \leq i \leq d-1 \} \\ &= \text{Span} \{ (E_0 + \dots + E_i) A^* - A^* (E_0 + \dots + E_i) \mid 0 \leq i \leq d-1 \} \\ &= \text{Span} \{ (E_0 + \dots + E_i) A^* - A^* (E_0 + \dots + E_i) \mid 0 \leq i \leq d \} \\ &= \{ ZA^* - A^*Z \mid Z \in M \} \end{aligned}$$

□

Notation

let λ, μ denote commuting indeterminates

- For $\beta, \gamma, \delta \in F$ define a polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

- For $\beta, \gamma^*, \delta^* \in F$ define

$$P^*(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma^*(\lambda + \mu) - \delta^*$$

() LEM 3† For $\beta, \gamma, \delta \in \mathbb{R}$ TFAE

$$(i) \quad 0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^*]$$

$$(ii) \quad P(\theta_i, \theta_i) = 0 \text{ for } 1 \leq i \leq d$$

pf Let $C = \text{RHS of (i)}$

$$\begin{aligned} C &= I C I \\ &= \sum_{i=0}^d \sum_{j=0}^d E_i C E_j \end{aligned}$$

For $0 \leq i, j \leq d$

$$\begin{aligned} E_i C E_j &= E_i A^* E_j (\theta_i - \theta_j) (\theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 - \gamma (\theta_i + \theta_j) - \delta) \\ &= E_i A^* E_j (\theta_i - \theta_j) P(\theta_i, \theta_j) \end{aligned}$$

(i) $\rightarrow$ (ii) $C = 0$ so for $1 \leq i \leq d$,

$$0 = E_{i-1} C E_i = \underbrace{E_{i-1} A^* E_i}_{\neq 0} \underbrace{(\theta_{i-1} - \theta_i)}_{\neq 0} P(\theta_{i-1}, \theta_i)$$

$$\text{so } P(\theta_{i-1}, \theta_i) = 0$$

(ii) $\rightarrow$ (i) Show $C = 0$ For $0 \leq i, j \leq d$ show

$$E_i C E_j = 0$$

$$E_i C E_j = E_i A^* E_j \underbrace{(a_i - a_j)}_{\substack{\parallel \\ 0 \\ \text{if } |i-j| > 1}} \underbrace{P(a_i, a_j)}_{\substack{\parallel \\ 0 \\ \text{if } |i-j| = 1}}$$

$$= 0$$

□

THM 38 $\exists \beta, \gamma, \gamma^*, \delta, \delta^* \in F$ st

$$0 = \left[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right]$$

"TD1"

$$0 = \left[A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A^* A + A A^*) - \delta^* A \right]$$

"TD2"

pf First assume $d \geq 3$.

By LEM 36 (with $X = A^2$, $Y = A$) $\exists Z \in M$ st

$$A^2 A^* A - A A^* A^2 = Z A^* - A^* Z \quad \star$$

By const $\{A^i\}_{i=0}^d$ form basis for M . $\exists$ poly f with degree $\leq d$

st

$$Z = f(A)$$

Define

$t = \text{degree } f$

$c = \text{leading coef of } f$

show $t=3$

Suppose $t > 3$.

Obs

$$0 = F_t \left(A^2 A^* A - A A^* A^2 - f(A) A^* + A^* f(A) \right) F_0$$

$$= \underbrace{c}_{\neq 0} \underbrace{(\theta_t^* - \theta_0^*)}_{\neq 0} \underbrace{R^t F_0}_{\neq 0} \neq 0 \quad \text{cont}$$

Suppose $t < 3$.

Obs

$$0 = F_3 \left(A^2 A^* A - A A^* A^2 - f(A) A^* + A^* f(A) \right) F_0$$

$$= \underbrace{(\theta_1^* - \theta_2^*)}_{\neq 0} \underbrace{R^3 F_0}_{\neq 0} \neq 0. \quad \text{cont}$$

So $t=3$.

Define

$$\beta = C^* - 1$$

So

$$C^* = \beta + 1$$

Divide $\star$ by c

this shows $\exists \gamma, \delta \in \mathbb{F}$ st

$$(\beta\gamma)(A^2A^*A - AA^*A^2) = A^3A^* - A^*A^3 - \gamma(A^2A^* - A^*A^2) \\ - \delta(AA^* - A^*A)$$

Rearrange terms to get

$$0 = \left[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^4 \right]$$

This is TD1.

To get TD2, pick i ($2 \leq i \leq d$). Mult each term in TD1 on left by F_{i-1} and right by F_{i-2} :

$$0 = \underbrace{\mathbb{R}^3 F_{i-2}}_{\neq 0} \underbrace{\left(\theta_{i-2}^* - (\beta\gamma)\theta_{i-1}^* + (\beta\gamma)\theta_i^* - \theta_{i+1}^* \right)}_{\text{must be 0}}$$

So $\{\theta_i^*\}_{i=0}^d$ is β -rec

So $\exists \gamma^*, \delta^* \in \mathbb{F}$ st $\{\theta_i^*\}_{i=0}^d$ is $(\beta, \gamma^*, \delta^*)$ -rec.

So $p^*(\theta_{i-1}^*, \theta_i^*) = 0 \quad (1 \leq i \leq d)$

Apply LEM3A to Φ^* ;

$$0 = \left[A^*, A^{*2}A - \beta A^*AA^* + A^{*2} - \gamma(A^*A + AA^*) - \delta^*A \right]$$

This is TD2

Done for $d \geq 3$

Next assume $d \leq 2$.

define

$\beta = \text{any scalar in } \mathbb{F}$

define

$$\gamma = \begin{cases} \theta_0 - \beta\theta_1 + \theta_2 & \text{if } d=2 \\ \text{arb} & \text{if } d \leq 1 \end{cases}$$

define

$$\delta = \begin{cases} \theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1) & \text{if } d=1 \\ \text{arb} & \text{if } d=0 \end{cases}$$

Then $\{\theta_i\}_{i=0}^d$ is (β, γ, δ) -rec, so TD1 holds.

Apply above argument to Φ^* ; $\exists \gamma^*, \delta^* \in \mathbb{F}$

st TD2 holds, □