

Math 846

Lecture 15

$\mathbb{F}$  arb $V = nm0$  fd vector space over  $\mathbb{F}$ Given TD system on  $V$ :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Recall

$$U_i = (E_0^* V + \dots + E_i^* V) \cap (E_0 V + \dots + E_{d-i} V) \quad 0 \leq i \leq d$$

Prop 30 The following hold for  $0 \leq i \leq j \leq d$ .

(i) the map

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ v & \longrightarrow & Q^{j-i} v \end{array} \quad (*)$$

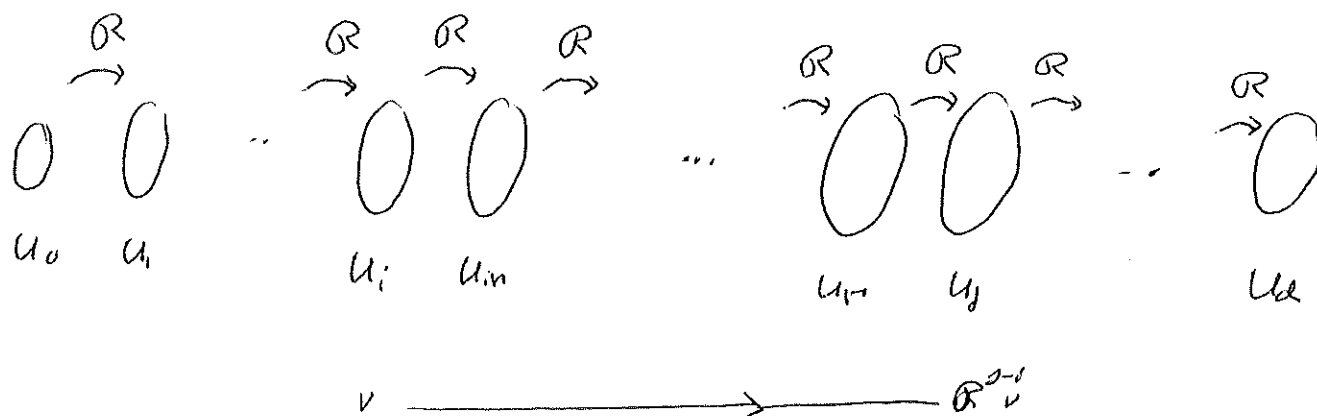
is injective if  $j \leq d-i$   
 big if  $j = d-i$   
 surj if  $j \geq d-i$

(ii) the map

$$\begin{array}{ccc} U_j & \longrightarrow & U_i \\ v & \longrightarrow & Q^{j-i} v \end{array} \quad (**)$$

is injective if  $j \geq d-i$   
 big if  $j = d-i$   
 surj if  $j \leq d-i$

(i) pf (i) We have



First assume  $j \leq d-i$

Given  $v \in U_i$  st  $R^{d-i} v = 0$ ,

show  $v = 0$ .

obs

$$\begin{aligned}
 0 &= R^{d-i} v \\
 &= \overbrace{R \ R \ \dots \ R}^{d-i} \underbrace{R v}_{\substack{\parallel \\ (A - \theta_{d-i} I) v \in U_{i+1}}}
 \end{aligned}$$

$$\underbrace{(A - \theta_{d-i-1} I) (A - \theta_{d-i} I) v}_{\parallel} \in U_{i+2}$$

$\vdots$

$$= (A - \theta_{d-j+1} I) \dots (A - \theta_{d-i} I) v$$

So

$$v \in E_{d-j+1}V + \dots + E_d V$$

$$\subseteq E_{d-j+1}V + \dots + E_d V$$

Also

$$v \in U_i = (E_0^*V + \dots + E_i^*V) \cap (E_0V + \dots + E_{d-j}V)$$

$$\subseteq E_0^*V + \dots + E_i^*V$$

So

$$v \in \underbrace{(E_0^*V + \dots + E_i^*V) \cap (E_dV + \dots + E_{d-j+1}V)}_{\parallel}$$

① since  $i+j-1 < d$ (this is Prop 17 applied to  $\Phi^{\vee}$ )

$$\text{So } v = 0 \quad \checkmark$$

So  $\pi$  is injective for  $j \leq d-i$  $\pi$  is big for  $j = d-i$  since  $\pi$  is inj and

$$\dim U_i = \dim U_j.$$

Next assume  $j \geq d-i$ show  $\pi$  surj.

By const

$$R^{j-i} U_i \subseteq U_j$$

Show

$$U_j \subseteq R^{j-i} U_i$$

By our previous comments

$$\begin{array}{ccc} U_{d-j} & \longrightarrow & U_j \\ v & \longrightarrow & R^{2j-d} v \end{array}$$

is big.

$$\begin{aligned} \text{So } U_j &= R^{2j-d} U_{d-j} \\ &= R^{j-i} \underbrace{R^{i+j-d}}_{\substack{\text{in} \\ U_i}} U_{d-j} \\ &\subseteq R^{j-i} U_i \end{aligned}$$

(ii) Similar

□

COR 31 The shape  $\{\rho_i\}_{i=0}^d$  of  $\Phi$  satisfies

$$\rho_{i-1} \leq \rho_i \quad (1 \leq i \leq d/2)$$

pf the map

$$U_{i-1} \rightarrow U_i$$

$$v \rightarrow \mathbb{R}v$$

is injective.

□

COR 32  $F_n$   $0 \leq i \leq j \leq d$ ,

$$\mathbb{R}^{1-i} F_i \neq 0,$$

$$\mathbb{L}^{1-i} F_j \neq 0$$

pf By Prop 30.

□

Next topic: the tetrahedron diagram of a tridiagonal system.

Recall the tridiagonal system on  $V$ :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

DEF 33 A decomposition of  $V$  is a

sequence  $\{W_i\}_{i=0}^d$  of non 0 subspaces of  $V$

such that

$$V = \sum_{i=0}^d W_i \quad (\text{direct sum})$$

For example

$$\{E_i V\}_{i=0}^d \quad \text{is a decomp of } V$$

$$\{E_i^* V\}_{i=0}^d \quad \dots$$

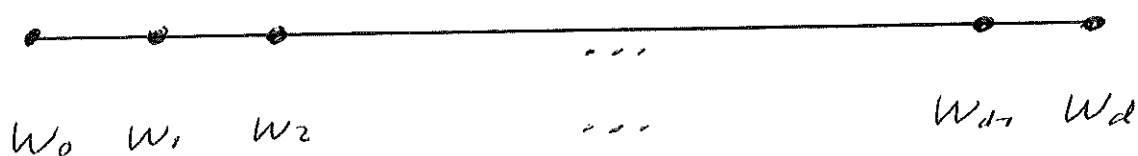
$$\{U_i\}_{i=0}^d \quad \dots$$

the shape of a decomp  $\{W_i\}_{i=0}^d$  is the sequence

$$\{\dim W_i\}_{i=0}^d$$

The inversion of a decomp  $\{W_i\}_{i=0}^d$  is the decomp  $\{W_{d-i}\}_{i=0}^d$

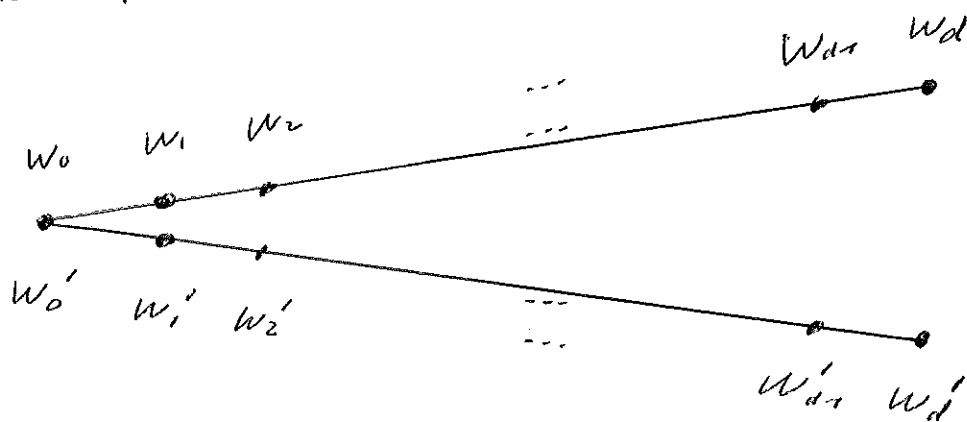
(1) We represent a decomp  $\{w_i\}_{i=0}^d$  of  $V$   
by a sequence of dots along a line segment:



Given two decompositions of  $V$ :

$$\{w_i\}_{i=0}^d \quad \{w'_i\}_{i=0}^d$$

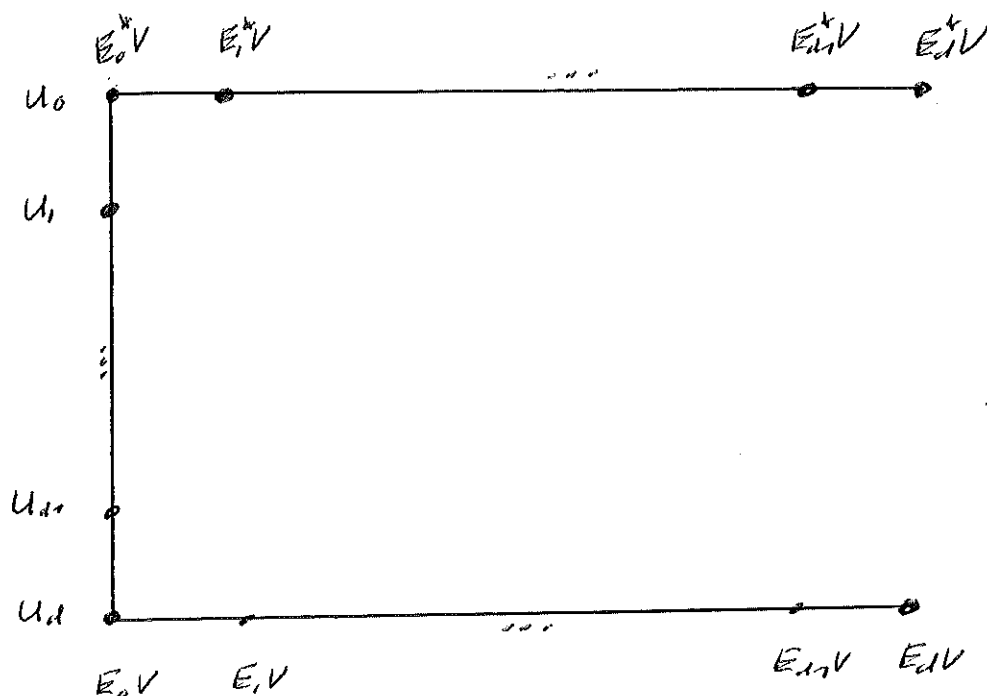
The diagram



means

$$w_0 + w_1 + \dots + w_d = w'_0 + w'_1 + \dots + w'_d \quad (0 \leq l \leq d)$$

By LEM 20,



The Northwest corner reminds us

$$U_0 + U_1 + \dots + U_{d-1} = E_0^* V + E_1^* V + \dots + E_{d-1}^* V \quad (0 \leq l \leq d)$$

The Southwest corner reminds us

$$U_d + U_{d-1} + \dots + U_l = E_0 V + E_1 V + \dots + E_{d-l} V \quad (0 \leq l \leq d)$$

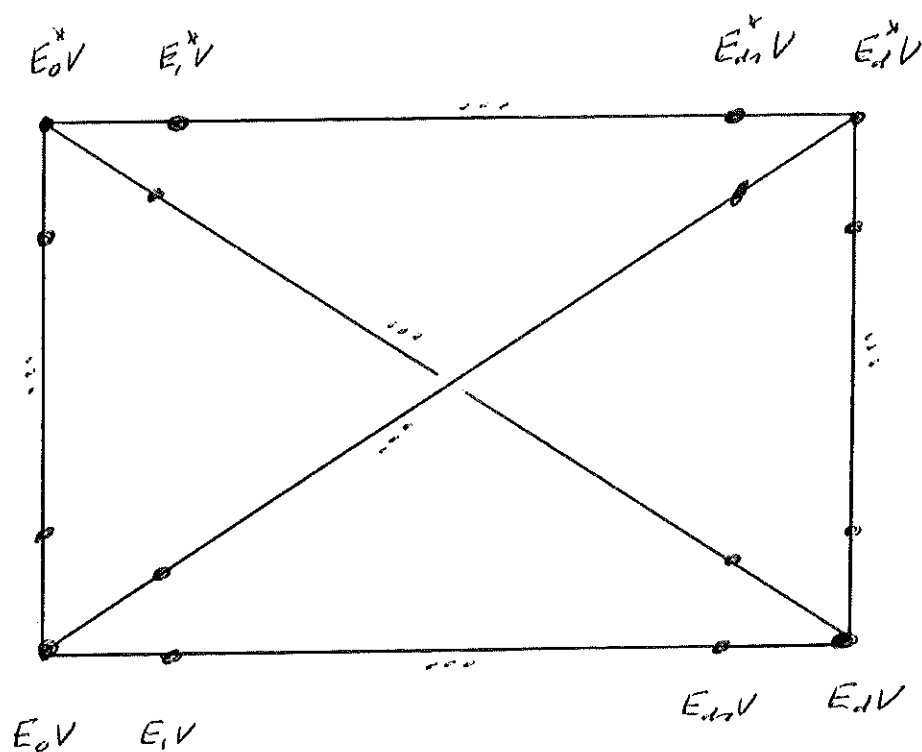
We call  $\{u_i\}_{i=0}^d$  the

$\Phi$ -split decomposition of  $V$

Next, we apply LEM 20 to the relations

$$\Phi^{\downarrow}, \quad \Phi^{\downarrow\downarrow}, \quad \Phi^{\downarrow\downarrow\downarrow}$$

We have



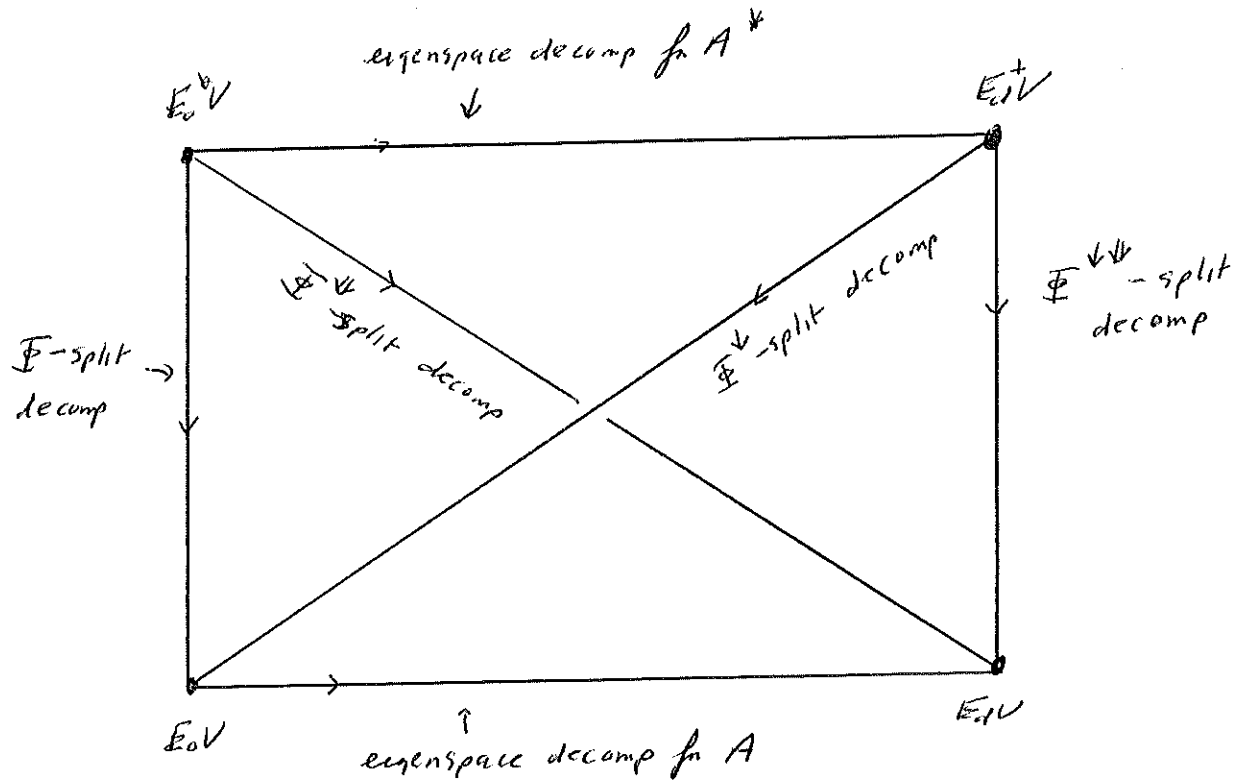
"tetrahedron diagram"

shows six decompositions of  $V$

Note the above decompositions all have

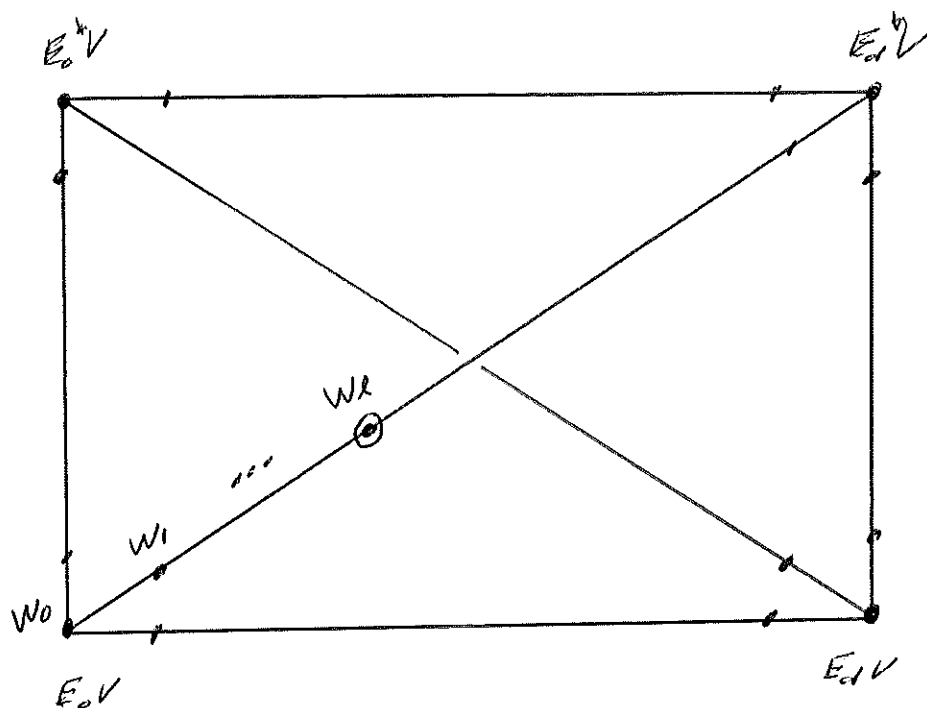
the same shape.

key:



The  $\Phi^*$ -split decomp of  $V$  is the inverse  
of the  $\Phi$ -split decomp of  $V$  etc.

EXAMPLE: Describe  $W_L$  in terms of  $\{E_i V\}_{i=0}^d$  and  $\{E_i^* V\}_{i=0}^d$



Sol:

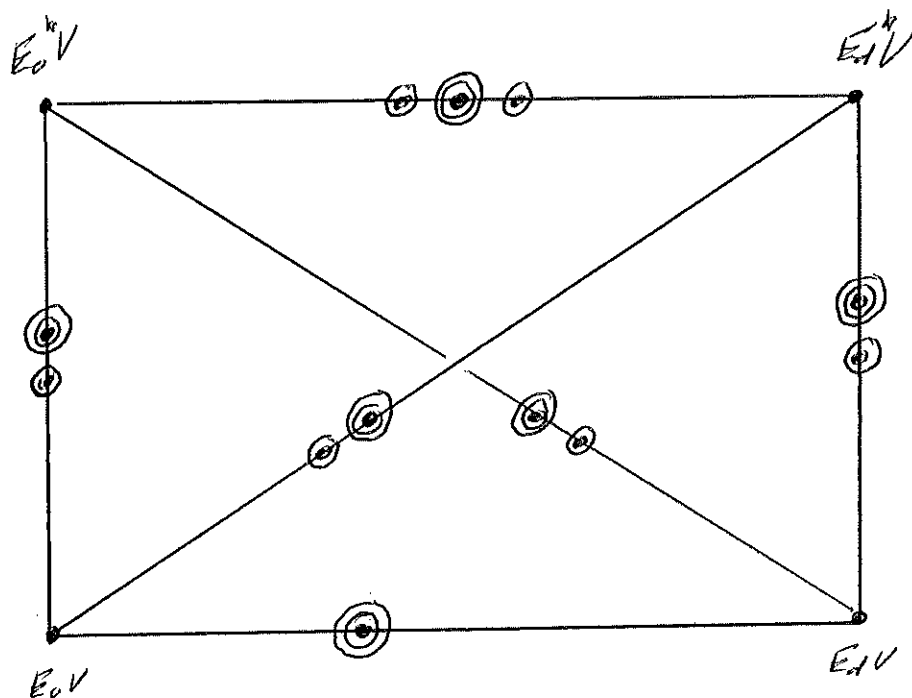
$$W_L = \underbrace{(w_0 + \dots + w_L)}_{E_0 V + \dots + E_d V} \cap \underbrace{(w_1 + \dots + w_d)}_{E_0^* V + \dots + E_d^* V}$$

So

$$W_L = (E_0 V + \dots + E_d V) \cap (E_0^* V + \dots + E_d^* V)$$

□

EXAMPLE: Describe the action of  $A$   
on each decomposition



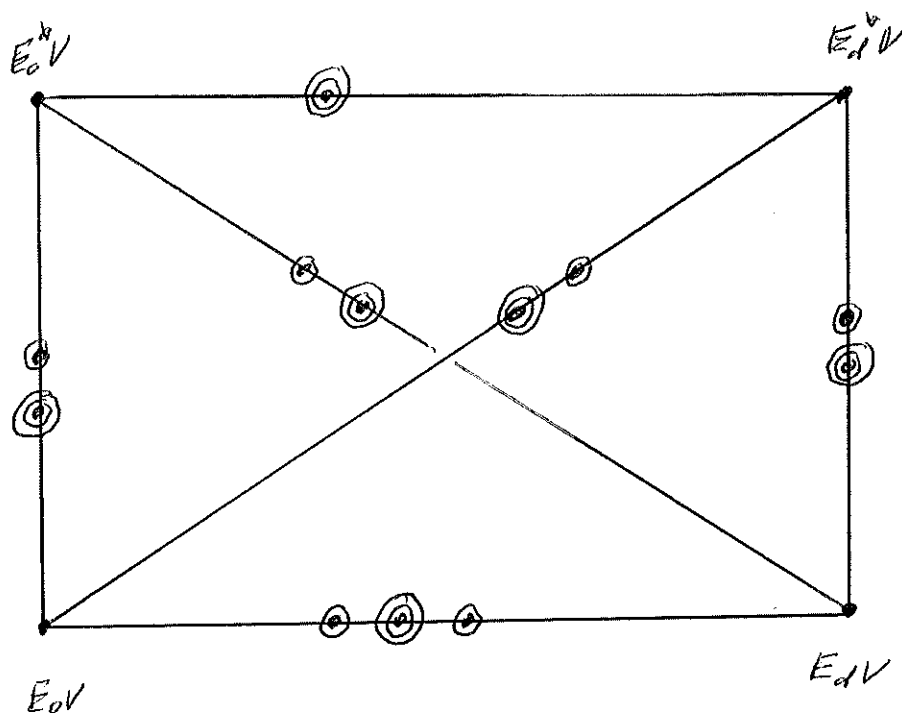
$A$  - action

key:

Large circles: input  $u$

small circles: their sum contains output  $Au$

EXAMPLE: Describe the action of  $A^*$   
on each decomposition



$A^*$ -action

key: Large circle: input  $U$

Small circles: their sum contains output  $A^*U$

We continue to discuss the TD system on  $V$ :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Our next general goal is to show

$$\bullet \quad \exists \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F} \text{ s.t.}$$

$$0 = \left[ A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right],$$

$$0 = \left[ A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \delta^* (A^* A + A A^*) - \delta^* A \right]$$

"

the tridiagonal relations

"

$$[R, S] = RS - SR$$

• The scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of  $i$  for  $2 \leq i \leq d-1$