

Math 846

Lecture 14

$\mathbb{F}$ arb

$V = \text{nontrivial vector space over } \mathbb{F}$

Given TD system in V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Recall

$$U_i = (E_0^* V + \dots + E_i^* V) \cap (E_0 V + \dots + E_{d-i} V) \quad 0 \leq i \leq d.$$

LEM 20 $F_n \quad 0 \leq i \leq d$

$$(i) \quad U_0 + \dots + U_i = E_0^* V + \dots + E_i^* V$$

$$(ii) \quad U_1 + \dots + U_d = E_0 V + \dots + E_{d-1} V$$

pf (i) $\subseteq F_n \quad 0 \leq i \leq d$

$$U_j = (E_0^* V + \dots + E_j^* V) \cap (E_0 V + \dots + E_{d-j} V)$$

$$\subseteq E_0^* V + \dots + E_j^* V$$

$$\subseteq E_0^* V + \dots + E_i^* V$$

2: Define

$$\psi = (A^* - \theta_d^* I) (A^* - \theta_{d-1}^* I) \cdots (A^* - \theta_{i+1}^* I)$$

obs

$$\psi V = E_0^* V + \cdots + E_i^* V$$

Show

$$\psi V \leq u_d + \cdots + u_i$$

Write

$$V = \sum_{j=0}^d u_j$$

(obvious)

Show

$$\psi u_j \leq u_d + \cdots + u_i$$

($0 \leq j \leq d$)

Case $0 \leq j \leq i$

$$\psi u_j \leq u_0 + \cdots + u_j$$

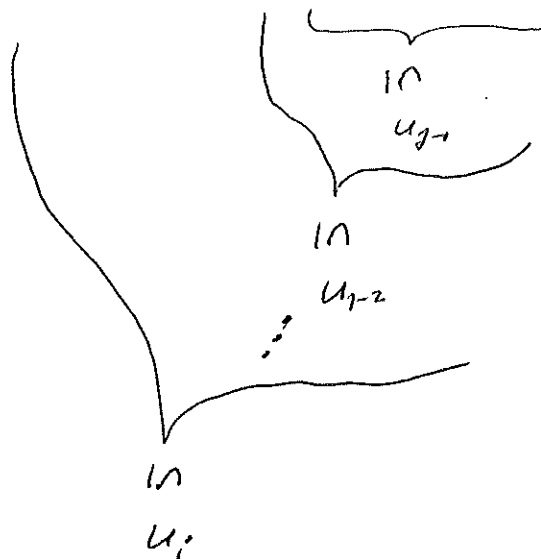
$$\leq u_0 + \cdots + u_i$$

(since $A^* u_l \leq u_l + u_{l+1} \forall l$)

Case $i+1 \leq j \leq d$

$$\psi U_j = (A^* - \theta_1^* I) \cdots (A^* - \theta_{j-1}^* I) (A^* - \theta_j^* I) \cdots (A^* - \theta_n^* I) U_j$$

$$= (A^* - \theta_1^* I) \cdots (A^* - \theta_{j-1}^* I) (A^* - \theta_j^* I) \cdots (A^* - \theta_n^* I) U_j$$



$$\subseteq U_0 + \cdots + U_n$$

$$\left(\text{since } A^* U_j \subseteq U_j + U_{j+1} \quad \forall j \right)$$

(ii) Similar

□

Next goal: For odd i show

$$E_i V, E_i^* V, U_i, E_{i+1} V, E_{i+1}^* V, U_{i+1}$$

all have same dimension.

(1) Recall the direct sum

$$V = \sum_{i=0}^d U_i$$

For $0 \leq i \leq d$ define

$$F_i \in \text{End}(V)$$

st

$$(F_i - I) U_i = 0$$

$$F_i U_j = 0 \text{ if } i \neq j \quad (0 \leq j \leq d)$$

Obs

$$F_i F_j = \delta_{ij} F_i \quad (0 \leq i, j \leq d)$$

$$I = \sum_{i=0}^d F_i$$

$$U_i = F_i V \quad (0 \leq i \leq d)$$

LEM 2) For $0 \leq i, j \leq d$

$$(i) \quad F_i E_j^* = 0 \quad \text{if } i > j$$

$$(ii) \quad E_i^* F_j = 0 \quad \text{if } i > j$$

$$(iii) \quad F_i E_j = 0 \quad \text{if } i+j < d$$

$$(iv) \quad E_i F_j = 0 \quad \text{if } i+j > d$$

p f (i) For $i > j$.

$$\begin{aligned} F_i E_j^* V &\leq F_i (E_0^* V + \dots + E_j^* V) \\ &= F_i (U_0 + \dots + U_j) \\ &= F_i (F_0 V + \dots + F_j V) \\ &= 0 \end{aligned}$$

(ii) Similar

(iii) For $i+j < d$

$$\begin{aligned} F_i E_j V &\leq F_i (E_0 V + \dots + E_j V) \\ &= F_i (U_0 + \dots + U_{d-j}) \\ &= F_i (F_0 V + \dots + F_{d-j} V) = 0 \end{aligned}$$

(iv) Sim.

□

LEM 22 For $0 \leq i \leq d$

$$(i) \quad F_i E_i^* F_i = F_i$$

$$(ii) \quad E_i^* F_i E_i^* = E_i^*$$

$$(iii) \quad F_i E_{d-i} F_i = F_i$$

$$(iv) \quad E_{d-i} F_i E_{d-i} = E_{d-i}$$

pf (i) Obs

$$\begin{aligned} F_i &= F_i^2 = F_i I F_i \\ &= \sum_{l=0}^d \underbrace{F_i E_l^* F_i}_{\substack{\text{if } i > l \\ 0}} \end{aligned}$$

$$= F_i E_i^* F_i$$

(iii)-(iv) Similar

□

LEM 23 For $0 \leq i \leq d$

(i) the maps

$$\begin{aligned} E_i^* V &\rightarrow U_i \\ v &\rightarrow F_i v \end{aligned}$$

$$\begin{aligned} U_i &\rightarrow E_i^* V \\ v &\rightarrow E_i^* v \end{aligned}$$

are bijections. Moreover they are inverses.

(ii) the maps

$$\begin{aligned} E_{d-i} V &\rightarrow U_i \\ v &\rightarrow F_i v \end{aligned}$$

$$\begin{aligned} U_i &\rightarrow E_{d-i} V \\ v &\rightarrow E_{d-i} v \end{aligned}$$

are bijections. Moreover they are inverses.

pf (i) check they are inverses:

$$E_i^* V \xrightarrow{F_i} U_i \xrightarrow{E_i^*} E_i^* V$$

$$v = E_i^* v \rightarrow F_i E_i^* v \rightarrow E_i^* F_i E_i^* v = E_i^* v = v$$

and

$$U_i \xrightarrow{E_i^*} E_i^* V \xrightarrow{F_i} U_i$$

$$v = F_i v \rightarrow E_i^* F_i v \rightarrow F_i E_i^* F_i v = F_i v = v$$

(ii) similar

□

LEM 24 For $0 \leq i \leq d$ the dimensions

$$E_i V, U_i, E_i^* V$$

are equal. Denoting this common dimension by

p_i , we have

$$p_i = p_{d-i}$$

pf By Lem 23,

$$\dim E_i^* V = \dim U_i = \dim E_{d-i} V$$

$$\dim E_{d-i}^* V = \dim U_{d-i} = \dim E_i V$$

Apply this to

$$\Phi^V = (A, \{E_{d-i}\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d);$$

$$\dim E_i^* V = \dim E_i V$$

$$\dim E_{d-i}^* V = \dim E_{d-i} V$$

Result follows.

□

DEF 25 the sequence $\{p_i\}_{i=0}^d$ is called the
shape of Φ .

Note that

$$\Phi, \Phi^*, \Phi^\downarrow, \Phi^{\downarrow*}$$

have the same shape.

Next goal: show the shape $\{p_i\}_{i=0}^d$ satisfies

$$p_{i+1} \leq p_i \quad 1 \leq i \leq d/2$$

" the shape is unimodal "

DEF 26 We define $R, L \in \text{End}(V)$ by

$$R = A - \sum_{i=0}^d \theta_{d-i} F_i$$

$$L = A^* - \sum_{i=0}^d \theta_i^* F_i$$

LEM 27 For $0 \leq i \leq d$ the following hold on U_i :

$$R = A - \theta_{d-i} I,$$

$$L = A^* - \theta_i^* I$$

pf Recall

$$F_i U_j = 0 \text{ if } i \neq j$$

LEM 28 We have

$$(i) \quad R U_i \subseteq U_{i-1} \quad (0 \leq i \leq d-1), \quad R U_d = 0$$

$$(ii) \quad L U_i \subseteq U_{i+1} \quad (1 \leq i \leq d), \quad L U_0 = 0$$

pf By LEM 14 and Lem 27.

□

We call R (resp. L) the raising map
(resp. lowering map) for Φ .

LEM 29 We have

$$(i) \quad R F_i = F_i R \quad (0 \leq i \leq d-1), \quad R F_d = 0, \quad F_0 R = 0$$

$$(ii) \quad L F_i = F_i L \quad (1 \leq i \leq d), \quad L F_0 = 0, \quad F_d L = 0$$

$$(iii) \quad R^{dn} = 0$$

$$(iv) \quad L^{dn} = 0$$

pf By Lem 28 and linear algebra

□

Prop 30 The following holds for $0 \leq i \leq j \leq d$.

(i) The map

$$U_i \longrightarrow U_j$$

$$v \longrightarrow R^{j-i} v$$

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is injective if $j \leq d-i$

bijective if $j = d-i$

surjective if $j \geq d-i$

(ii) The map

$$U_j \longrightarrow U_i$$

$$v \longrightarrow L^{j-i} v$$

**

is injective if $j \geq d-i$

bijective if $j = d-i$

surjective if $j \leq d-i$

(The above maps are not inverses in general, even
for $i+j=d$)