

Math 846

Lecture 13

$\mathbb{F}$ arb

$V = nm \times d$ vector space over $\mathbb{F}$

TD pair A, A^* on V

An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called standard whenever

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d)$$

$$V_{-1} = 0, \quad V_{d+1} = 0$$

If $\{V_i\}_{i=0}^d$ is standard then the inverted ordering

$\{V_{d-i}\}_{i=0}^d$ is standard, and no further ordering is standard.

A similar discussion applies to the eigenspaces of A^* .

For A or A^* , an ordering of the primitive idempotents is standard whenever the corresp ordering of the eigenspaces is standard.

DEF 11 A tridiagonal system on V (TD system)

is a sequence

$$\mathcal{F} = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

such that

(i) A, A^* is a TD pair on V

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*

Obs Given a TD system in V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

then the following are TD systems in V :

$$\Phi^* = (A^*, \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$$

$$\Phi^\downarrow = (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$$

$$\Phi^\Downarrow = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

Viewing $*$, $\downarrow$, $\Downarrow$ as permutations on the set of all TD systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = \text{id}$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow \Downarrow = \Downarrow \downarrow$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the above relations is the dihedral group D_4

Recall D_4 is the group of symmetries of a square, and has 8 elements.

So $\ast, \downarrow, \Downarrow$ induce a D_4 -action on the set of all TD systems.

Two TD systems are relatives whenever they are in the same orbit of this D_4 -action.

Two TD systems are relatives whenever their associated TD pair is equal or dual.

Until further notice, we fix a TD system in V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^{\delta})$$

Next goal: show $d = \delta$.

LEM 12 We have

$$(i) \quad E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leq i, j \leq d)$$

$$(ii) \quad E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leq i, j \leq d)$$

pf (i) For $|i-j| > 1$,

$$E_i A^* E_j = 0 \quad \text{by Def 3 (ii)}$$

For $|i-j| = 1$,

$$E_i A^* E_j \neq 0 \quad \text{by Def 3 (iv)}$$

(ii) Similar

□

DEF 13 We refer to Φ .

For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for E_i .

Call $\{\theta_i\}_{i=0}^d$ the eigenvalue sequence of Φ .

For $0 \leq i \leq d$ let θ_i^* denote the eigenvalue of A^* for E_i^* .

Call $\{\theta_i^*\}_{i=0}^d$ the dual eigenvalue sequence of Φ .

We emphasize the $\{\theta_i\}_{i=0}^d$ are not distinct
and $\{\theta_i^*\}_{i=0}^d$ are not distinct.

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For $0 \leq i \leq d$ and $0 \leq j \leq d$ define

$$V_{ij} = (E_0^* V + E_1^* V + \dots + E_i^* V) \cap (E_0 V + E_1 V + \dots + E_j V)$$

$$\text{So } V_{0,d} = E_0^* V,$$

$$V_{d,0} = E_0 V$$

Define

$$N = \min \{i+j \mid V_{ij} \neq 0\}$$

obs

$$N \leq d \quad \text{since } V_{0,d} \neq 0$$

$$N \leq \delta \quad \text{since } V_{\delta,0} \neq 0$$

For $0 \leq i \leq N$ write

$$U_i = V_{i, N-i}$$

so

$$U_i = (E_0^* V + E_1^* V + \dots + E_i^* V) \cap (E_0 V + E_1 V + \dots + E_{N-i} V) \quad (\star)$$

shortly we will show $N = d = \delta$

LEM 14 We have

$$(i) \quad (A^* - \theta_i^* I) U_i \leq U_{i+1} \quad (1 \leq i \leq N)$$

$$(ii) \quad (A^* - \theta_0^* I) U_0 = 0$$

$$(iii) \quad (A - \theta_{N-i} I) U_i \leq U_{i+1} \quad (0 \leq i \leq N-1)$$

$$(iv) \quad (A - \theta_0 I) U_N = 0$$

pf (i) obs

$$(A^* - \theta_i^* I) (E_0^* V + \dots + E_i^* V) = E_0^* V + \dots + E_{i+1}^* V$$

and

$$(A^* - \theta_i^* I) (E_0 V + \dots + E_{N-i} V) \leq E_0 V + \dots + E_{N-i+1} V$$

(ii) Set $i=0$ in U_i to get

$$U_0 = E_0^* V$$

(iii), (iv) Similar

□

Cor 15 We have

$$(i) \quad A^* u_i \leq u_i + u_i \quad (1 \leq i \leq N)$$

$$(ii) \quad A^* u_0 \leq u_0$$

$$(iii) \quad A u_i \leq u_i + u_{i+1} \quad (0 \leq i \leq N-1)$$

$$(iv) \quad A u_N \leq u_N$$

□

pf By Lem 14.

$$\text{Lem 16} \quad V = \sum_{i=0}^N u_i$$

$$\text{pf Write } u = \sum_{i=0}^N u_i$$

$u \neq 0$ since at least one $u_i \neq 0$

By Cor 15,

$$A u \leq u, \quad A^* u \leq u.$$

So $u = V$ by Def 3 (iv)

□

Prop 17 $N = d = \delta$

pf Recall $N \leq d, \quad N \leq \delta$

By $\star$, for $0 \leq i \leq N$ we have

$$\begin{aligned} u_i &\leq E_0 V + \dots + E_{N-i} V \\ &\leq E_0 V + \dots + E_N V \end{aligned}$$

so

$$\begin{aligned} E_0 V + \dots + E_d V &= V \\ &= u_0 + \dots + u_N \\ &\leq E_0 V + \dots + E_N V \end{aligned}$$

so $N = d$.

By $\star$, for $0 \leq i \leq N$ we have

$$\begin{aligned} u_i &\leq E_0^* V + \dots + E_i^* V \\ &\leq E_0^* V + \dots + E_N^* V \end{aligned}$$

so

$$\begin{aligned} E_0^* V + \dots + E_\delta^* V &= V \\ &= u_0 + \dots + u_N \\ &\leq E_0^* V + \dots + E_N^* V \end{aligned}$$

so $N = \delta$

□

DEF 18 We saw $d = \delta$; call this the
diameter of $\mathbb{F}$.

Call $\mathbb{F}$ trivial whenever $\mathbb{F}$ has diameter 0

Obs $\mathbb{F}$ is trivial iff $\dim V = 1$

LEM 19 The sum

$$V = \sum_{i=0}^d U_i$$

is direct.

pf Suffices to show

$$0 = (U_0 + U_1 + \dots + U_{i-1}) \cap U_i \quad (1 \leq i \leq d)$$

Obs $(U_0 + \dots + U_{i-1}) \cap U_i \subseteq (E_0^* V + \dots + E_{i-1}^* V) \cap (E_0 V + \dots + E_d V)$

$$= V_{i-1, d-i}$$

$$= 0 \quad \text{since } i-1 + d-i < d$$

□

Next goal: show $\forall 0 \leq i \leq d$

$$\bullet \quad U_0 + U_1 + \dots + U_i = E_0^* V + E_1^* V + \dots + E_i^* V$$

$$\bullet \quad U_i + U_{i+1} + \dots + U_d = E_0^* V + E_1^* V + \dots + E_{d-i}^* V$$