

Math 846

Lecture 12

We continue to discuss the algebra

U_q^+ over $\mathbb{F}$ $\mathbb{F}$ arb,

$$0 \neq q \in \mathbb{F} \quad q^2 \neq 1$$

DEF 7 Let V denote a finite dim'l irred U_q^+ -module. Call V nonnil whenever neither of A, A^* is nilpotent on V

$$\text{i.e.} \quad A^n \neq 0, \quad A^{*n} \neq 0 \quad \forall n \geq 1$$

DEF 8 We say q is not a root of unity

whenever

$$q^n \neq 1 \quad \forall n \geq 1.$$

EXAMPLE 9 Assume $\mathbb{F}$ alg closed. Pick $0 \neq q \in \mathbb{F}$ that is not a root of unity.

Let V denote a fd irred U_q^+ -module that is nonnil. Then A, A^* act on V as a tridiagonal pair.

pf For $\theta \in \mathbb{F}$ define

$$V(\theta) = \{v \in V \mid Av = \theta v\}$$

claim

$$A^* V(\theta) \subseteq V(q^{-2}\theta) + V(\theta) + V(q^2\theta) \quad \text{if } \theta \in \mathbb{F}$$

pf cl Use

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0$$

Put $v \in V(\theta)$. So $Av = \theta v$. We have

$$\begin{aligned} 0 &= [A, [A, [A, A^*]_q]_{q^{-1}}] v \\ &= \left(A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 \right) v \\ &= \left(A^3 - [3]_q A^2 \theta + [3]_q A \theta^2 - \theta^3 \right) A^* v \\ &= (A - q^{-2}\theta)(A - \theta)(A - q^2\theta) A^* v \end{aligned}$$

The scalars $q^{-2}\theta, \theta, q^2\theta$ are mut distinct so

$$A^* v \in V(q^{-2}\theta) + V(\theta) + V(q^2\theta)$$

claim proved

□ Since F is alg closed and A is not nilpotent
 $\exists$ non eigenvalue θ for A .

Since q not a root of unity,

$$\theta, q^{-2}\theta, q^{-4}\theta, \dots$$

are mut distinct, so not all eigenvalues for A .

So $\exists$ non eigenvalue α of A st $q^{-2}\alpha$ is
 not an eigenvalue of A .

Similarly

$$\alpha, q^2\alpha, q^4\alpha, \dots$$

mut dist, so not all eigenvalues of A .

So $\exists d \in \mathbb{N}$ st

αq^{2i} is an eigenvalue of A ($0 \leq i \leq d$)

αq^{2d+2} is not an eigenvalue of A

Abbrev

$$V_i = V(q^{2i}\alpha) \quad 0 \leq i \leq d$$

By constr

$$V_0 + V_1 + \dots + V_d$$

is non 0 and A -invar

By the claim,

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$$

(0 $\leq i \leq d$)

where $V_{-1} = 0$ and $V_{d+1} = 0$

Thus

$$V_0 + V_1 + \dots + V_d$$

is invar under A, A^*

So

$$V = V_0 + \dots + V_d$$

by used of V_0

this shows A is diagonalizable on V and
Def 3 (ii) holds.

Interchanging roles of A, A^* we find A^* is
diagonalizable on V and Def 3 (iii) holds.

Note Def 3 (iv) holds since U_q^+ -module V is used. $\square$

Next goal: clarify how Leonard pairs and tridiagonal pairs are related.

$\mathbb{F}$ arb

let $V = n \times m$ f.d. vector space over $\mathbb{F}$

A map $A \in \text{End}(V)$ is called multiplicity free

whenever

- A is diagonalizable
- each eigenspace of A has dimension 1

LEM 10 For $A, A^* \in \text{End}(V)$ TFAE

(i) A, A^* is Leonard pair on V

(ii) A, A^* is a tridiagonal pair on V st each of A, A^* is mult free

Pf (i) $\rightarrow$ (ii) $\exists$ basis $\{v_i\}_{i=0}^d$ of V

wrt which A is diagonal and A^* is irred TD.

Denote these representing matrices by B and B^* resp

From the pattern of zero/nonzero entries the

matrices $\{(B^*)^i\}_{i=0}^d$ are lin indep.

$\therefore \{(A^*)^i\}_{i=0}^d$ are lin indep.

$\therefore$ min poly of A^* has degree $d+1 = \dim V$

$\therefore A^*$ has $d+1$ eigenvalues.

Each eigenspace of A^* must have $\dim \geq 1$

A^* mult free

Similarly, A mult free

For $0 \leq i \leq d$ define

$$V_i = \text{span}(v_i)$$

By const

V_i is eigenspace of A

$$A^k V_i \subseteq V_{i-1} + V_i + V_{i+1}$$

($0 \leq i \leq d$)

where $V_{-1} = 0, V_{d+1} = 0$

Similarly $\exists$ ordering $\{v_i^*\}_{i=0}^d$ of the eigenspaces of A^*

st

$$A v_i^* \subseteq v_{i-1}^* + v_i^* + v_{i+1}^* \quad (0 \leq i \leq d)$$

$$v_{-1}^* = 0, \quad v_{d+1}^* = 0.$$

Given subspace $0 \neq W \subseteq V$ st

$$AW \subseteq W$$

$$A^*W \subseteq W.$$

show $W = V$

Since $AW \subseteq W \quad \exists$ subset $S \subseteq \{0, 1, \dots, d\}$

$$\text{st} \quad W = \sum_{i \in S} v_i^*$$

Since B^* is irred TD

$$i \in S \text{ and } |i-j|=1 \text{ implies } j \in S \quad (0 \leq i, j \leq d)$$

$$\text{so} \quad S = \{0, 1, \dots, d\}$$

so

$$W = V$$

therefore A, A^* is TD pair on V

(iii) $\rightarrow$ (i) $\exists$ ordering $\{v_i\}_{i=0}^d$ of the eigenspaces of A st

$$A^* v_i \subseteq v_{i-1} + v_i + v_{i+1} \quad (0 \leq i \leq d)$$

$$v_{-1} = 0, \quad v_{d+1} = 0$$

Since A is mult free

$$\dim v_i = 1 \quad (0 \leq i \leq d)$$

Pick

$$0 \neq v_i \in v_i \quad (0 \leq i \leq d)$$

So $\{v_i\}_{i=0}^d$ is basis for V

Matrix B^* rep A^* rel $\{v_i\}_{i=0}^d$ is TD

TD matrix B^* must be unid, else

$$\exists W \subseteq V \text{ st}$$

$$W \neq 0, \quad W \neq V, \quad AW \subseteq W, \quad A^*W \subseteq W.$$

Therefore rel $\{v_i\}_{i=0}^d$ the matrix rep A is diagonal and the matrix rep A^* is irred TD.

Similarly $\exists$ basis for V with resp to which A is irred TD and A^* is diagonal.

So A, A^* is LP on V

□

$\mathbb{F}$ arb

Let A, A^* denote a tridiagonal pair (TD pair)

on V .

Then A^*, A is a TD pair on V "the dual of A, A^* "

Also, given $r, s, r^*, s^* \in \mathbb{F}$ st

$$r \neq 0, \quad r^* \neq 0$$

the pair

$$rA + sI, \quad r^*A^* + s^*I$$

is a TD pair on V .

"affine transformations of A, A^* "

Next goal: introduce the concept of a tridiagonal system.

Notation Given diagonalizable $A \in \text{End}(V)$

Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A

For $0 \leq i \leq d$ let $\theta_i = \text{eigenvalue of } A \text{ for } V_i$

For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ s.t.

$$(E_i - I)V_i = 0$$

$$E_i V_j = 0 \quad \text{if } j \neq i \quad (0 \leq j \leq d)$$

So E_i is primitive idempotent of A for V_i

Recall

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d)$$

$$I = \sum_{i=0}^d E_i$$

$$A = \sum_{i=0}^d \theta_i E_i$$

$$V_i = E_i V \quad (0 \leq i \leq d)$$

$$A E_i = \theta_i E_i = E_i A \quad (0 \leq i \leq d)$$

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d)$$

Let $M =$ subalgebra of $\text{End}(V)$ generated by A .

The elements $\{A^i\}_{i=0}^d$ form basis for M , and

$$0 = \prod_{i=0}^d (A - a_i I)$$

Also, the elements $\{E_i\}_{i=0}^d$ form basis for M .