

Math 846

Lecture 11

II Leonard pairs and tridiagonal pairs

Until further notice $\mathbb{F} = \text{any field, possibly finite.}$

Consider any square matrix A over $\mathbb{F}$

$A:$

$$\begin{pmatrix} & & & * \\ & & & \\ & & & \\ * & & & \end{pmatrix}$$

$\nwarrow$ superdiagonal
 $\swarrow$ diagonal
 $\nwarrow$ subdiagonal

Call A tridiagonal whenever each non 0 entry is on the diagonal, the superdiagonal, or the subdiagonal.

Assume A is tridiagonal. Call A irreducible whenever each entry on superdiagonal is non 0 and each entry on subdiagonal is non 0

Let V denote a non 0 vector space over F with finite dimension.

Let $\text{End}(V)$ denote the F -algebra of all F -linear maps from V to V .

DEF 1 By a Leonard pair on V we mean an ordered pair $A, A^* \in \text{End}(V)$ st:

- (i) $\exists$ basis for V with respect to which the matrix rep A is unred tridiagonal and the matrix rep A^* is diagonal.
- (ii) $\exists$ basis for V with respect to which the matrix rep A^* is unred tridiagonal and the matrix rep A is diagonal.

Example 2 Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

let $\Gamma = H(0, 2)$

Fix $x \in X$ write $T = T(x)$

let W denote an irred T -module with diameter d .

We saw:

In one basis

$$A: \begin{pmatrix} 0 & d & & & 0 \\ 1 & 0 & d-1 & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ & & & d & 0 \end{pmatrix}$$

$$A^*: \text{diag}(d, d-2, d-4, \dots, -d)$$

In another basis

$$A: \text{diag}(d, d-2, d-4, \dots, -d)$$

$$A^*: \begin{pmatrix} 0 & d & & & 0 \\ 1 & 0 & d-1 & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ & & & d & 0 \end{pmatrix}$$

So the pair A, A^* acts on W as a Leonard pair.

We will classify the Leonard pairs.

First, we consider something more general called a tridiagonal pair.

$\mathbb{F}$ a field

Assume V is a nonzero vector space over $\mathbb{F}$ with finite dimension.

For $A \in \text{End}(V)$ call A diagonalizable

whenever V is spanned by the eigenspaces of A ,

DEF 3 By a tridiagonal pair on V , we mean an ordered pair $A, A^* \in \text{End}(V)$ st

(i) each of A, A^* is diagonalizable

(ii) $\exists$ ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A st

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$

(iii) $\exists$ ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* st

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta)$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$

(iv) $\nexists$ subspace $W \subseteq V$ st

$AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$

EXAMPLE 4 Given a distance-regular graph

$\Gamma = (X, E)$ with adjacency matrix A

Fix $x \in X$ and write $T = T(x)$

Assume Γ is $\mathbb{Q}$ -polynomial with dual adjacency matrix $A^* = A^*(x)$.

Let W denote an irred. T -module.

Then A, A^* act on W as a tridiagonal pair.

Pf By discussion after notation 61.

□

EXAMPLE 5 Assume $\mathbb{F}$ is alg closed, char 0

Let V denote a finite dim'l mod $\mathcal{O}$ -module.

Then the $\mathcal{O}$ -generators A, A^* act on V as a tridiagonal pair.

pt For $\theta \in \mathbb{F}$ define

$$V(\theta) = \{v \in V \mid Av = \theta v\}$$

obs $V(\theta) \neq 0$ iff θ is an eigenvalue for A on V , and

in this case $V(\theta)$ is corresp. eigenspace.

We claim

$$A^*V(\theta) \subseteq V(\theta-2) + V(\theta) + V(\theta+2) \quad (\theta \in \mathbb{F})$$

pt cl Use

$$[A, [A, [A, A^*]]] = 4[A, A^*]$$

Pick $v \in V(\theta)$ so $Av = \theta v$

obs

$$0 = \left([A, [A, [A, A^*]]] - 4[A, A^*] \right) v$$

$$= \left(A^3 A^* - 3A^2 A^* A + 3A A^* A^2 - A^* A^3 - 4(A A^* - A^* A) \right) v$$

$$= \left(A^3 - 3A^2 \theta + 3A \theta^2 - \theta^3 - 4(A - \theta) \right) A^* v$$

$$= \left((A - \theta)^3 - 4(A - \theta) \right) A^* v$$

$$= (A - \theta - 2)(A - \theta)(A - \theta + 2) A^* v$$

Note $\theta - 2, \theta, \theta + 2$ are distinct, so

$$A^* v \in V(\theta - 2) + V(\theta) + V(\theta + 2)$$

claim proved ✓

Since $\mathbb{F}$ is alg closed, $\exists$ at least one eigenvalue θ of A on V .

Since $\mathbb{F}$ is char 0, the scalars

$$0, 0-2, 0-4, \dots$$

are mutually distinct, so they are not all eigenvalues of A on V .

So $\exists$ eigenvalue α of A on V s.t. $\alpha-2$ is not an eigenvalue of A on V .

Similarly,

$$\alpha, \alpha+2, \alpha+4, \dots$$

are mutually distinct, so not all eigenvalues of A on V .

So $\exists d \in \mathbb{N}$ s.t.

$\alpha + 2i$ is eigenvalue of A on V ($0 \leq i \leq d$)

$\alpha + 2d+2$ is not an eigenvalue of A on V .

Abbrev

$$V_i = V(\alpha + 2i)$$

$$0 \leq i \leq d$$

then by constr

$$V_0 \oplus V_1 \oplus \dots \oplus V_d$$

is non 0 and A -invariant.

By the claim,

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d).$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

Thus

$$V_0 + V_1 + \dots + V_d$$

is invariant for both A, A^*

So,

$$V = V_0 + V_1 + \dots + V_d$$

by the irreducibility of V .

This shows that A is diagonalizable on V
and Def 3 (ii) holds.

Interchanging $A \leftrightarrow A^*$ we find A^* is diagonalizable
on V and Def 3 (iii) holds.

Note Def 3 (iv) holds since $\mathcal{O}$ -module V is
irreducible. □

$\mathbb{F}$ arb

Given $0 \neq q \in \mathbb{F}$ st $q^2 \neq 1$

define
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}$$

For elements R, S in any algebra, define

$$[R, S] = RS - SR \quad \text{"commutator"}$$

$$[R, S]_q = qRS - q^{-1}SR \quad \text{"q-commutator"}$$

Obs

$$[R, [R, [R, S]_q]_{q^{-1}}] =$$

$$R^3S - [3]_q R^2SR + [3]_q RSR^2 - SR^3$$

DEF 6 For $0 \neq q \in \mathbb{F}$ st $q^2 \neq 1$, define

an associative $\mathbb{F}$ -algebra $U_q^+ = U_q^+(\mathbb{F})$

by generators and relations, as follows.

The gens are A, A^*

The relations are

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$

$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0$$

" q -Serre relations "

Call U_q^+ the positive part of $U_q(\mathfrak{sl}_2)$