

Math 846

Lecture 10

Aside on Lie algebras

For this aside assume

 $\mathbb{F}$ = any field with $\text{char } \mathbb{F} = 0$

Def 74 A Lie algebra over $\mathbb{F}$ is a
vector space L over $\mathbb{F}$ together with a

map $[\cdot, \cdot] : L \times L \rightarrow L$ "Lie bracket"
 $u, v \mapsto [u, v]$

s.t.

$$(i) \quad [\alpha u, v] = \alpha [u, v]$$

$$\alpha \in \mathbb{F} \quad u, v \in L$$

$$(ii) \quad [u + u', v] = [u, v] + [u', v]$$

$$u, u', v \in L$$

$$(iii) \quad [u, \alpha v] = \alpha [u, v]$$

$$\alpha \in \mathbb{F} \quad u, v \in L$$

$$(iv) \quad [u, v + v'] = [u, v] + [u, v']$$

$$u, v, v' \in L$$

$$(v) \quad [u, u] = 0$$

$$u \in L$$

$$(vi) \quad \forall u, v, w \in L$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

"Jacobi identity"

Example 75 For $n \geq 2$ let $\mathfrak{sl}_n(\mathbb{F})$ denote the Lie algebra consisting of the $n \times n$ matrices with entries in $\mathbb{F}$ and trace 0, together with Lie bracket

$$[u, v] = uv - vu$$

Example 76 the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and Lie bracket

$$[h, e] = 2e$$

$$[h, f] = -2f$$

$$[e, f] = h$$

Given Lie algebras L and L' over $\mathbb{F}$.

A Lie algebra homomorphism from L to L'

is an $\mathbb{F}$ -linear map $\sigma: L \rightarrow L'$ st

$$\sigma([u, v]) = [\sigma(u), \sigma(v)] \quad \forall u, v \in L$$

A Lie algebra isomorphism from L to L'

is a bijective Lie alg hom from L to L' .

We say L, L' are isomorphic whenever

$\exists$ Lie alg iso from L to L' .

DEF 77 Given a Lie algebra L over F

An L module is a vector space V over F

together with a map

$$L \times V \rightarrow V$$

$$u, v \rightarrow u \cdot v$$

st

$$(i) \quad (\alpha u) \cdot v = \alpha(u \cdot v)$$

$$\alpha \in F \quad u \in L \quad v \in V$$

$$(ii) \quad (u + u') \cdot v = u \cdot v + u' \cdot v$$

$$u, u' \in L \quad v \in V$$

$$(iii) \quad u \cdot (\alpha v) = \alpha(u \cdot v)$$

$$\alpha \in F \quad u \in L \quad v \in V$$

$$(iv) \quad u \cdot (v + v') = u \cdot v + u \cdot v'$$

$$u \in L \quad v, v' \in V$$

$$(v) \quad [u, u'] \cdot v = u \cdot (u' \cdot v) - u' \cdot (u \cdot v)$$

$$u, u' \in L \quad v \in V$$

DEF 78 Given a Lie algebra L over F

An L -module V is called irreducible whenever $V \neq 0$ and V does not contain an L -submodule besides 0 and V .

EX 79 Let V denote a finite-dim'l irred module for $sl_2(F)$. Then V has a basis

$$v_i \quad (0 \leq i \leq d)$$

st

$$h \cdot v_i = (d - 2i) v_i \quad (0 \leq i \leq d)$$

$$f \cdot v_i = (i+1) v_{i+1} \quad (0 \leq i \leq d-1), \quad f \cdot v_d = 0$$

$$e \cdot v_i = (d-i+1) v_{i-1} \quad (1 \leq i \leq d), \quad e \cdot v_0 = 0$$

pf: Mimic what we did for hypercube

LEM 80 Assume $F = \mathbb{R}$ or $\mathbb{C}$

let $\Gamma = H(0, 2)$

Fix $x \in X$ and write $T = T(x)$

Then the standard module V of Γ becomes
an $\mathfrak{sl}_2(F)$ -module on which e, f, h
act as follows.

element u	e	f	h
action of u on V	L	R	A^*

pf Because

$$A^*L - LA^* = 2L$$

$$A^*R - RA^* = -2R$$

$$LR - RL = A^*$$

matches the relations in $\mathfrak{sl}_2(F)$ for e, f, h

□

Def 81 Define the associative $\mathbb{F}$ -algebra $U(\mathfrak{sl}_2(\mathbb{F}))$ by generators and relations, as follows.

The generators are $E, F, H,$

The relations are

$$HE - EH = 2E$$

$$HF - FH = -2F$$

$$EF - FE = H$$

$U(\mathfrak{sl}_2(\mathbb{F}))$ is called the universal enveloping algebra

of $\mathfrak{sl}_2(\mathbb{F})$.

LEMMA Assume $F = \mathbb{R}$ or $\mathbb{C}$

$$\text{let } T = H(0, 2)$$

Fix $x \in X$ and write $T = T(x)$.

then $\exists$ surjective algebra homomorphism

$$U(\mathfrak{sl}_2(F)) \longrightarrow T$$

that sends

$$E \rightarrow L$$

$$F \rightarrow R$$

$$H \rightarrow A^*$$

pf Because

$$A^*L - LA^* = 2L,$$

$$A^*R - RA^* = -2R$$

$$LR - RL = A^*$$

matches the defining relations for $U(\mathfrak{sl}_2(F))$

□

Another view of $\mathfrak{sl}_2(\mathbb{F})$

Recall

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

define

$$a = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$a^* = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So

$$\begin{aligned} [a, a^*] &= -[a^*, a] \\ &= -[h, e+f] \\ &= -2e + 2f \\ &= 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

 $\mathfrak{sl}_2(\mathbb{F})$ has basis

$$a, \quad a^*, \quad [a, a^*]$$

and

$$[a, [a, a^*]] = 4a^*, \quad [a^*, [a^*, a]] = 4a$$

Another view of $U(\mathfrak{sl}_2(\mathbb{F}))$

Define the associative algebra $\mathcal{L}$ by generators and relations, as follows.

The gens are A, A^* .

The relations are

$$\begin{aligned} A^2 A^* - 2 A A^* A + A^* A^2 &= 4 A^* \\ A^* A^2 - 2 A^* A A^* + A A^{*2} &= 4 A \end{aligned} \quad \left(\begin{array}{l} \text{compare with} \\ \text{M52} \end{array} \right)$$

LEM 83 $\exists$ $\mathbb{F}$ -algebra isomorphism $\sigma: \mathcal{L} \rightarrow U(\mathfrak{sl}_2(\mathbb{F}))$

σ sends

$$A \rightarrow E + F$$

$$A^* \rightarrow H$$

The inverse iso sends

$$E \rightarrow \frac{A^* A - A A^* + 2 A}{4},$$

$$F \rightarrow \frac{A A^* - A^* A + 2 A}{4}$$

$$H \rightarrow A^*$$

pf Similar to Lem 51 and Mm 52

□

The Onsager Lie algebra $\mathcal{O}$

Field $\mathbb{F}$ char 0

Motivation start with $\mathfrak{sl}_2(\mathbb{F})$

basis $a, a^*, [a, a^*]$

and $[a, [a, a^*]] = 4a^*, \quad [a^*, [a^*, a]] = 4a$

We have

$$[a, [a, [a, a^*]]] = 4[a, a^*] \quad (DG1)$$

$$[a^*, [a^*, [a^*, a]]] = 4[a^*, a] \quad (DG2)$$

Call these the Dolan/Grady relations

DEF 84 We define a Lie algebra $\mathcal{O}$ over $\mathbb{F}$ by generators and relations. The gens are A, A^*

The relations are

$$[A, [A, [A, A^*]]] = 4[A, A^*]$$

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]$$

Call $\mathcal{O}$ the Onsager Lie algebra

Ex 85

The Onsager Lie algebra $\mathcal{O}$ has a

basis

$$A_i, B_i, C_i \quad i \in \mathbb{N}$$

where $A_0 = A$ and $B_0 = A^*$ and for $i \in \mathbb{N}$

$$[A_i, A_j] = 0$$

$$[B_i, B_j] = 0$$

$$[C_i, C_j] = 0$$

$$[C_i, A_j] = 2C_{i+j} + 2A_{i+j}$$

$$[B_i, C_j] = 2B_{i+j} + 2C_{i+j}$$

$$[A_i, B_j] = 2A_{i+j} + 2B_{i+j} - 4C_{i+j}$$

By constr $\exists$ Lie algebra hom $\theta \rightarrow \mathfrak{sl}_2(\mathbb{F})$

that sends

$$A \rightarrow a, \quad A^* \rightarrow a^*$$

DEF 86 Define an assoc $\mathbb{F}$ -algebra $U(\theta)$

by generators and relations as follows.

The gens are A, A^*

The relations are

$$[A, [A, [A, A^*]]] = 4[A, A^*]$$

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]$$

where $[R, S]$ means $RS - SR$

Call $U(\theta)$ the universal enveloping algebra of θ

By constr. $\exists$ $\mathbb{F}$ -alg hom $U(\theta) \rightarrow U(\mathfrak{sl}_2(\mathbb{F}))$

that sends

$$A \rightarrow A \quad A^* \rightarrow A^*$$