

Math 846

Lecture 1

I the hypercube $H(0,2)$ and the
Lie algebra sl_2

Preliminaries

Natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$

Integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

Field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (for now)

Let $X =$ nonempty, finite set (of vertices)

$V = \mathbb{F}^X$ is the vector space over $\mathbb{F}$

consisting of column vectors with coordinates

indexed by X and entries in $\mathbb{F}$.

For $x \in X$ define vector

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x$$

The vectors

$$\{\hat{x} \mid x \in X\}$$

form a basis for V .

We endow V with a form

$$\langle u, v \rangle = u^t \overset{\text{transpose}}{\overline{v}} \quad (u, v \in V)$$

complex conjugate

$\langle, \rangle$ is Hermitian if $F = \mathbb{C}$ and symmetric if $F = \mathbb{R}$

The basis $\{\hat{x} \mid x \in X\}$ is orthonormal wrt $\langle, \rangle$

Define $\text{Mat}_X(F)$ to be the set of

matrices that have rows/columns indexed by X

and all entries in F . $\text{Mat}_X(F)$ becomes an

algebra under matrix addition, matrix mult,
and scalar mult.

$\text{Mat}_X(\mathbb{F})$ acts on V by left multiplication.

Call V the standard module

$\text{Mat}_X(\mathbb{F})$ contains

- identity matrix I
- all 1's matrix J

A graph is an ordered pair $\Gamma = (X, E)$

such that

- X is a nonempty finite set "the vertices"
- E is a set of 2-element subsets of X , "the edges"

Vertices $x, y \in X$ are called adjacent whenever $\{x, y\} \in E$

Our graph Γ is

- finite
- undirected
- no loops
- no multiple edges

For $x \in X$ define

$\Gamma(x)$ = set of vertices that are adjacent to x .

For $n \in \mathbb{N}$, a walk of length n in Γ is a sequence of vertices

$$(x_0, x_1, \dots, x_n)$$

(*)

such that

$$x_i \in \Gamma(x_{i-1}) \quad \text{for } 1 \leq i \leq n$$

We say the walk (*) is from x_0 to x_n

A path of length n in Γ is a walk $(x_0, x_1, \dots, x_n)$

in Γ s.t. $x_{i-1} \neq x_{i+1}$ ($1 \leq i \leq n-1$) "no backtrack allowed"

Define a binary rel $\sim$ on X s.t.

$x \sim y$ whenever $\exists$ walk in Γ from x to y ($x, y \in X$)

Note $\sim$ is an equiv relation

the equivalence classes of $\sim$ are called the connected components of Γ .

Γ is connected whenever X is a connected component

From now on, we assume Γ is connected unless otherwise stated.

For $x, y \in X$ define

$$d(x, y) = \min \left\{ n \mid n \in \mathbb{N}, \exists \text{ walk of length } n \text{ from } x \text{ to } y \right\}$$

"the distance between x and y "

define

$$D = \max \left\{ d(x, y) \mid x, y \in X \right\}$$

"the diameter of Γ "

For $x \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(x) = \left\{ y \in X \mid d(x, y) = i \right\}$$

" i th subconstituent of Γ wrt x "

So $\Gamma_0(x) = \{x\}$

$$\Gamma_1(x) = \Gamma(x)$$

An automorphism of Γ is a permutation of X st

$$x, y \text{ adj} \iff \sigma(x), \sigma(y) \text{ adj} \\ (x, y \in X)$$

The set of automorphisms of Γ form a group under composition; call the group $\text{Aut}(\Gamma)$

Note for $x, y \in X$ and $\sigma \in \text{Aut}(\Gamma)$,

$$d(x, y) = d(\sigma x, \sigma y)$$

Γ is called vertex-transitive whenever

$$\forall x, y \in X \quad \exists \sigma \in \text{Aut}(\Gamma) \text{ st } \sigma(x) = y$$

Γ is called edge-transitive whenever

$\forall$ adjacent x, y and adjacent x', y'

$$\exists \sigma \in \text{Aut}(\Gamma) \text{ st}$$

$$\sigma(x) = x' \text{ and } \sigma(y) = y'$$

Γ is called distance-transitive whenever

$$\forall x, y, x', y' \in X \text{ st } d(x, y) = d(x', y')$$

$$\exists \sigma \in \text{Aut}(\Gamma) \text{ st}$$

$$\sigma(x) = x' \text{ and } \sigma(y) = y'$$

Define $A \in \text{Mat}_X(\mathbb{F})$ with (x, y) -entry

$$A_{x,y} = \begin{cases} 1 & \text{if } x, y \text{ adjacent} \\ 0 & \text{if } x, y \text{ not adjacent} \end{cases} \quad x, y \in X$$

"adjacency matrix of Γ "

Note for all $x \in X$,

$$A x^\top = \sum_{y \in \Gamma(x)} y^\top$$

Lemma 1 $\forall x, y \in X$ and $l \in \mathbb{N}$

$$(A^l)_{x,y} = \text{number of walks of length } l \text{ from } x \text{ to } y$$

pf Using matrix mult

$$(A^l)_{x,y} = \underbrace{(A A A \dots A)}_l_{x,y}$$

$$= \sum_{x_1 \in X} \sum_{x_2 \in X} \dots \sum_{x_{l+1} \in X} A_{x,x_1} A_{x_1,x_2} \dots A_{x_{l+1},y}$$

$\nwarrow \uparrow \dots \nearrow$

0 unless indices are adjacent

$$= \sum 1$$

$\uparrow$ sum over all the walks of length l from x to y

□

LEM 2 The matrices

$$I, A, A^2, \dots, A^D$$

($D = \text{diameter of } \Gamma^2$)

are linearly independent.

pf Suppose not.

$$\exists l \ (1 \leq l \leq D) \text{ st}$$

$$A^l \in \text{span} \{ A^i \mid 0 \leq i \leq l-1 \}$$

$$\exists x, y \in X \text{ st } d(x, y) = l$$

$$(A^l)_{x,y} \neq 0$$

$$(A^i)_{x,y} = 0 \quad (0 \leq i \leq l-1)$$

Contradiction.

□

Def 3 Let M denote the subalgebra of $\text{Mat}_X(F)$ generated by A . So

$$M = \text{span} \{ A^l \mid l \in \mathbb{N} \}.$$

Call M the adjacency algebra or Bose/McCner algebra of Γ .

Note by Lem 2 that

$$\dim M \geq D+1$$

— 0 —

Since A is real and symmetric, it is diagonalizable.

By an eigenvalue of Γ we mean a root of the minimal polynomial of A .

For an eigenvalue θ of Γ let

$$\text{mult}(\theta) = \text{dimension of the } \theta\text{-eigenspace of } A$$

"the multiplicity of θ "

Note that

$$\begin{aligned} \# \text{ eigenvalues of } P &= \text{degree of min poly of } A \\ &= \dim M \\ &\geq D+1 \end{aligned}$$

Unless we state otherwise, the eigenvalues are denoted

$$\theta_0 > \theta_1 > \theta_2 > \dots$$

Write

$$m_i = \text{mult}(\theta_i)$$

Def 4 For $D \geq 1$ the hypercube $H(D, 2)$ (n -cube)

satisfies:

- Vertex set X consists of the D -tuples $(a_1, a_2, \dots, a_D)$ st $a_i \in \{1, -1\}$ ($1 \leq i \leq D$), so $|X| = 2^D$.
- Vertices $x, y \in X$ are adjacent whenever x, y differ in exactly one coordinate.

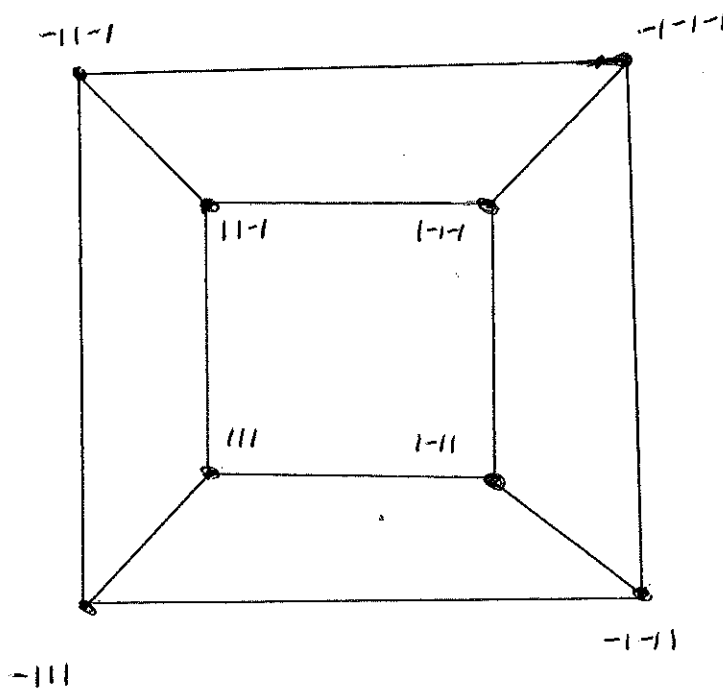
Note for $H(D, 2)$

$$\partial(x, y) = \# \text{ coordinates at which } x, y \text{ differ. } (x, y \in X)$$

Moreover

$H(D, 2)$ has diameter D .

For $D=3$



Next goal: And the eigenvalues of $H(D, 2)$

First consider small D

$$D=1$$

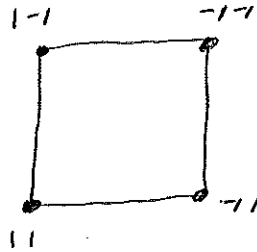
$P:$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalue	1	-1
mult	1	1

$$D=2$$

 $\vec{v}$

 $A =$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 11 \\ 1-1 \\ -1-1 \\ -11 \end{matrix}$$

 $11 \quad 1-1 \quad -1-1 \quad -11$

$\text{rank}(A) = 2$ so 0 is an eigenvalue with mult 2

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

2 is eigenvalue

$$A \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

-2 is eigenvalue

eigenvalue	2	0	-2
mult	1	2	1