2- Homogeous bipartite distance-regular graphs

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3. The eigenvalues and dual eigenvalues

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LEM 2 Let Γ denote a bipartite DRG with diameter D and valency k.

Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of Γ.

Then $\theta_i = -\theta_{D-i}$ $(0 \le i \le D)$

$\mathrm{mult}(\theta_i) = \mathrm{mult}(\theta_{D-i})$ $(0 \le i \le D)$.

pf LEM 3 in Ch 1. $\square$

DEF 3 Let Γ denote a bipartite DRG with valency k.

We say that an eigenval θ of Γ is trivial if $\theta = k$ or $\theta = -k$

LEM 4 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam $D \ge 1$.

Pick any $\theta, \underbrace{\theta_0^*, \theta_1^*, \ldots, \theta_D^*}_{\text{not all zero}} \in \mathbb{C}$, and set $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$.

Then TFAE:

(i) θ is an eigenval of Γ, and E is the associated primitive idempotent

(ii) $\forall x, y \in X$, $\langle E\hat{x}, E\hat{y} \rangle = \theta_i^* |X|^{-1}$ where $i = \partial(x, y)$,

and $\sum_{z \in \Gamma_1(x)} E\hat{z} = \theta E\hat{x}$

(iii) The intersection numbers of Γ satisfy

$$\theta\theta_i^* = c_i \theta_{i-1}^* + b_i \theta_{i+1}^* \quad (0 \le i \le D)$$

where θ_{-1}^*, θ_{D+1}^* denote indeterminants, and $\theta_0^* = \mathrm{rank}(E)$

If (i)–(iii) hold, then $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are real.

In this case, we call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the dual eigenvalue sequence associated with θ (or E).

Pf (i) => (ii)

$\langle \bar{E}\hat{x}, \bar{E}\hat{y} \rangle = \langle \bar{E}^t \bar{E}\hat{x}, \hat{y} \rangle = \langle \bar{E}\hat{x}, \hat{y} \rangle = \bar{E}_{xy} = \theta_i^* |X|^{-1}$ where $i = \partial(x,y)$ ✓

$\theta \bar{E}\hat{x} = A\bar{E}\hat{x} = \bar{E}A\hat{x} = \bar{E}\sum_{z\in\Gamma_1(x)} \hat{z} = \sum_{z\in\Gamma_1(x)} \bar{E}\hat{z}$ ✓

(ii) => (iii)

$\forall\, x,y$ st $\partial(x,y) = i$,

$$\left\langle \sum_{z\in\Gamma_1(x)} \bar{E}\hat{z}, \bar{E}\hat{y} \right\rangle = \sum_{z\in\Gamma_1(x)} \langle \bar{E}\hat{z}, \bar{E}\hat{y} \rangle = \sum_{z\in\Gamma_1(x)} \theta^*_{\partial(z,y)} |X|^{-1} = |X|^{-1} \sum_{j=0}^{D} \overbrace{|\Gamma(x)\cap\Gamma_j(y)|}^{p^i_{1j}} \theta_j^* = \underline{|X|^{-1}(c_i\theta^*_{i-1} + b_i\theta^*_{i+1})}$$

$\parallel$

$\langle \theta\bar{E}\hat{x}, \bar{E}\hat{y} \rangle = |X|^{-1}\underline{\theta\theta_i^*}$ ✓

$\langle \bar{E}\hat{x}, \bar{E}\hat{y} \rangle = (\bar{E}\hat{x})^t \bar{E}\hat{y} = \hat{x}^t \bar{E}^2 \hat{y} = \bar{E}^2_{xy}$

$\parallel$

$\theta_i^* |X|^{-1} = \bar{E}_{xy}$

so $\bar{E}^2 = \bar{E}$

so $\operatorname{rank}(\bar{E}) = \operatorname{tr}(\bar{E}) = \theta_0^*$ ✓

(iii) => (i)

$$\underline{A\bar{E}} = A\left(|X|^{-1}\sum_{i=0}^{D}\theta_i^* A_i\right) = |X|^{-1}\sum_{i=0}^{D}\theta_i^* \underbrace{AA_i} = \theta|X|^{-1}\sum_{i=0}^{D}\theta_i^* A_i = \underline{\theta\bar{E}} \checkmark$$

LEM 7 in Ch 2 $\rightarrow$ $= b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$, $a_i = 0$

$$= \sum_{i=0}^{D} \theta_i^* (b_{i-1}A_{i-1} + c_{i+1}A_{i+1})$$

$$= \sum_{i=-1}^{D-1} \theta^*_{i+1} b_i A_i + \sum_{i=1}^{D} \theta^*_{i-1} c_i A_i$$

$A_{-1} = 0,\ b_0 = 0,\ c_0 = 0 \Rightarrow$

$$= \sum_{i=0}^{D} \theta^*_{i+1} b_i A_i + \sum_{i=0}^{D} \theta^*_{i-1} c_i A_i$$

$$= \sum_{i=0}^{D} \theta\theta_i^* A_i$$

□

LEM 5 Let $\Gamma = (X, R)$ denote bipartite DRG with diam D.

Let θ denote an eigenval of Γ.

Let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq.

Then $\theta_0^* \geq \theta_i^* \geq -\theta_0^*$ $(0 \leq i \leq D)$

Pf Pick $x, y \in X$ st $\partial(x,y) = i$.

LEM 4 (ii) gives that $\underbrace{\frac{1}{2}|X| \, \| E\hat{x} - E\hat{y} \|^2}_{\geq 0} = \theta_0^* - \theta_i^*$

$$\underbrace{\frac{1}{2}|X| \, \| E\hat{x} + E\hat{y} \|^2}_{\geq 0} = \theta_0^* + \theta_i^*$$

□

LEM 6 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam D and valency $k \geq 2$.

Let θ denote an eigenval of Γ, and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq.

(i) If $D \geq 1$, then $\dfrac{\theta_1^*}{\theta_0^*} = \dfrac{\theta}{k}$.

(ii) If $D \geq 2$, then $\dfrac{\theta_2^*}{\theta_0^*} = \dfrac{\theta^2 - k}{k(k-1)}$

(iii) If $D \geq 3$, then $\dfrac{\theta_3^*}{\theta_0^*} = \dfrac{\theta(\theta^2 + \mu - k - k\mu)}{k(k-1)(k-\mu)}$ $\quad (\mu := c_2)$

Pf Recall LEM 4 (iii)

(i) Set $i = 0$ and Recall $c_0 = 0$, $b_0 = k$.

(ii) Use (i) and set $i = 1$. Recall $c_1 = 1$, $b_1 = k - c_1 = k - 1$.

(iii) Use (i), (ii) and set $i = 2$.

COR 7 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam $D \geq 2$ and valency $k \geq 3$.

Let θ denote a nontrivial eigenval of Γ, and let $\theta_0^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq.

(i) $\theta_0^* > \theta_2^* > -\theta_0^*$

(ii) $E\hat{x}$ and $E\hat{y}$ are linearly independent for all $x, y \in X$ with $\partial(x,y) = 2$

pf (i) Use LEM 5, LEM 6 (ii) and since θ is nontrivial.

(ii) Pick $x, y \in X$ with $\partial(x,y) = 2$.

$$\operatorname{Det}\begin{pmatrix} \langle E\hat{x}, E\hat{x}\rangle & \langle E\hat{x}, E\hat{y}\rangle \\ \langle E\hat{y}, E\hat{x}\rangle & \langle E\hat{y}, E\hat{y}\rangle \end{pmatrix} \underset{\text{LEM 4 (ii)}}{=} |X|^{-2}(\theta_0^* - \theta_2^*) \underset{\text{(i)}}{\neq} 0$$

so the matrix of inner products for $E\hat{x}$ and $E\hat{y}$ is nonsingular.

so $E\hat{x}$ and $E\hat{y}$ are linearly independent. □

LEM 8 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam $D \geq 2$ and valency $k \geq 2$.
Let θ denote a nontrivial eigenval of Γ, and let $\theta_0^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq.

(i) $\theta_1^* \theta_D^* = \theta_0^* \theta_{D-1}^*$

(ii) Fix i $(1 \leq i \leq D)$, and assume that $\theta_{i-1}^* \neq \theta_{i+1}^*$. Then $c_i = \dfrac{\theta\theta_i^* - k\theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*}$

(iii) $k = \dfrac{\theta_0^*(\theta_0^* - \theta_2^*)}{\theta_1^{*2} - \theta_0^*\theta_2^*}$

pf (i) Set $i = 0$ and $i = D$ in LEM 4 (ii). Recall $b_0 = k$, $c_D = k$.

(ii) Set $b_i = k - c_i$ in LEM 4 (ii), and solve for c_i

(iii) Set $i = 1$, $c_1 = 1$ in (ii), eliminate θ using LEM 6 (i), and solve for k.

LEM 9 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam D.
Let θ denote an eigenval of Γ, and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq.
Then the dual eigenval seq associated with $-\theta$ is $\{(-1)^i\theta_i^* \mid 0 \leq i \leq D\}$.

pf LEM 4 (ii) gives $\begin{cases} \theta\theta_i^* = c_i\theta_{i-1}^* + b_i\theta_{i+1}^* \iff -\theta(-1)^i\theta_i^* = c_i(-1)^{i-1}\theta_{i-1}^* + b_i(-1)^{i+1}\theta_{i+1}^* \\ \theta_0^* = \operatorname{rank}(E) \end{cases}$

LEM 2 gives that $-\theta$ is an eigenval and let E' denote the associated primitive idempotent

$$(-1)^0\theta_0^* = \theta_0^* = \operatorname{rank}(E) = \operatorname{mult}(\theta) = \operatorname{mult}(-\theta) = \operatorname{rank}(E')$$

LEM 10 Let $\Gamma = (X, R)$ denote a bipartite DRG with diam ≥ 1. Let θ denote an eigenval of Γ. Let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenval seq. Then TFAE:

(i) θ is the second largest eigenval of Γ

(ii) $\theta_0^* > \theta_1^* > \cdots > \theta_D^*$

In particular, if (i) and (ii) hold, then $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are distinct.

Pf This is a special case of the following LEM.

Section 13.
LEM 2.1 (Reference: C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993)

Let $w_0, \ldots, w_d$ be a seq of cosines for a DRG, belonging to the eigenval θ. Assume that θ is the ith largest eigenval of A. Then the cosine seq $w_0, \ldots, w_d$ has exactly $i-1$ sign-changes

if $i \geq 2$, the seq $w_0 - w_1, \ldots, w_d - w_{d+1}$ has exactly $i-2$ sign-changes.