Leonard triples of q-Racah type

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Overview

This talk concerns a Leonard triple A, B, C of q-Racah type.

We will describe this triple, using three invertible linear maps called W, W', W''.

As we will see,

- A commutes with W and $W^{-1}BW - C$;
- B commutes with W' and $(W')^{-1}CW' - A$;
- C commutes with W'' and $(W'')^{-1}AW'' - B$.

Moreover the three elements $W'W, W''W', WW''$ mutually commute, and their product is a scalar multiple of the identity.

Motivation: Leonard pairs

Before describing a Leonard triple, we first describe a more basic object called a Leonard pair.

We will use the following notation.

Let $\mathbb{F}$ denote a field.

Fix an integer $d \geq 0$.

Let V denote a vector space over $\mathbb{F}$ with dimension $d+1$.

Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps from V to V.

The Definition of a Leonard Pair

Definition (Terwilliger 1999)

By a **Leonard pair** on V, we mean an ordered pair of maps in $\mathrm{End}(V)$ such that for each map, there exists a basis of V with respect to which the matrix representing that map is diagonal and the matrix representing the other map is irreducible tridiagonal.

So for a Leonard pair A, B

	A	B
basis 1	diagonal	irred. tridiagonal
basis 2	irred. tridiagonal	diagonal

The term **Leonard pair** is motivated by a 1982 theorem of **Doug Leonard** concerning the q-Racah polynomials and some related polynomials in the Askey scheme.

For a detailed version of Leonard's theorem see the book

E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin Cummings. London 1984.

We mention one feature of Leonard pairs.

By an **antiautomorphism** of $\mathrm{End}(V)$ we mean an $\mathbb{F}$-linear bijection $\dagger : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $(XY)^\dagger = Y^\dagger X^\dagger$ for all $X, Y \in \mathrm{End}(V)$.

Lemma (Terwilliger 2001)

Let A, B denote a Leonard pair on V. Then there exists a unique antiautomorphism $\dagger$ of $\mathrm{End}(V)$ that fixes each of A, B.

Leonard triples

The notion of a Leonard triple is due to **Brian Curtin** and defined as follows.

Definition (Brian Curtin 2006)

By a **Leonard triple** on V, we mean a 3-tuple of maps in $\mathrm{End}(V)$ such that for each map, there exists a basis of V with respect to which the matrix representing that map is diagonal and the matrices representing the other two maps are irreducible tridiagonal.

Leonard triples in summary

So for a Leonard triple A, B, C

	A	B	C
basis 1	diagonal	irred. tridiagonal	irred. tridiagonal
basis 2	irred. tridiagonal	diagonal	irred. tridiagonal
basis 3	irred. tridiagonal	irred. tridiagonal	diagonal

Note that any two of A, B, C form a Leonard pair on V.

The above bases will be called **standard**.

The eigenvalues

Let A, B, C denote a Leonard triple on V.

By construction each of A, B, C is diagonalizable, and it turns out that their eigenspaces all have dimension 1.

Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A.

This ordering is called **standard** whenever a corresponding eigenbasis of A is standard.

Assume that the ordering $\{\theta_i\}_{i=0}^d$ is standard.

Then the inverted ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard. Similar comments apply to B and C.

Modular Leonard triples

The study of Leonard triples began with Curtin's comprehensive treatment of a special case, said to be modular.

Definition (Curtin 2006)

A Leonard triple on V is called **modular** whenever for each element of the triple there exists an antiautomorphism of $\mathrm{End}(V)$ that fixes that element and swaps the other two elements of the triple.

In 2006 Curtin classified up to isomorphism the modular Leonard triples.

General Leonard triples

Recently the general Leonard triples have been classified up to isomorphism, via the following approach.

Using the eigenvalues one breaks down the analysis into four special cases, called q-Racah, Racah, Krawtchouk, and Bannai/Ito.

The Leonard triples are classified up to isomorphism by

- Hau-wen Huang 2012 (for q-Racah type);
- Sougang Gao, Y. Wang, Bo Hou 2013 (for Racah type);
- N. Kang, Bou Hou, Sougang Gao 2015 (for Krawtchouk type);
- Bo Hou, L. Wang, Sougang Gao, Y. Xu 2013, 2015 (for Bannai/Ito type).

An example of a Leonard triple

We now describe the Leonard triples of q-Racah type (following Hau-wen Huang).

Fix nonzero scalars a, b, c, q in $\mathbb{F}$ such that $q^4 \neq 1$.

From now on assume:

(i) $q^{2i} \neq 1$ for $1 \leq i \leq d$;

(ii) None of a^2, b^2, c^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$;

(iii) None of abc, $a^{-1}bc$, $ab^{-1}c$, abc^{-1} is among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

Leonard triple example, cont.

For $0 \leq i \leq d$ define

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i},$$
$$\theta'_i = bq^{2i-d} + b^{-1}q^{d-2i},$$
$$\theta''_i = cq^{2i-d} + c^{-1}q^{d-2i}.$$

Note that for $0 \leq i, j \leq d$,

$$\theta_i \neq \theta_j, \qquad \theta'_i \neq \theta'_j, \qquad \theta''_i \neq \theta''_j \qquad \text{if} \quad i \neq j.$$

Leonard triple example, cont.

For $1 \leq i \leq d$ define

$$\varphi_i = a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1}) \\ \times \quad (q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1}).$$

Note that $\varphi_i \neq 0$.

Define

$$
A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}
$$

$$
B = \begin{pmatrix} \theta'_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta'_1 & \varphi_2 & & & \\ & & \theta'_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta'_d \end{pmatrix}
$$

A Leonard triple example, cont.

Theorem (Hau-wen Huang 2011)

For the above A, B there exists an element C such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(a+a^{-1})(q^{d+1}+q^{-d-1}) + (b+b^{-1})(c+c^{-1})}{q+q^{-1}} I,$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(b+b^{-1})(q^{d+1}+q^{-d-1}) + (c+c^{-1})(a+a^{-1})}{q+q^{-1}} I,$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(c+c^{-1})(q^{d+1}+q^{-d-1}) + (a+a^{-1})(b+b^{-1})}{q+q^{-1}} I.$$

The above equations are called the $\mathbb{Z}_3$-**symmetric Askey-Wilson relations**.

A Leonard triple example, cont.

Theorem (Hau-wen Huang 2011)

(i) *The above* A, B, C *form a Leonard triple on* $V = \mathbb{F}^{d+1}$.

(ii) $\{\theta_i\}_{i=0}^d$ *is a standard ordering of the eigenvalues of* A;

(iii) $\{\theta'_i\}_{i=0}^d$ *is a standard ordering of the eigenvalues of* B;

(iv) $\{\theta''_i\}_{i=0}^d$ *is a standard ordering of the eigenvalues of* C.

The above Leonard triple is said to have q-**Racah type**, with **Huang data** (a, b, c, d).

Notation

For notational convenience define

$$\alpha_a = \frac{(a+a^{-1})(q^{d+1}+q^{-d-1})+(b+b^{-1})(c+c^{-1})}{q+q^{-1}},$$

$$\alpha_b = \frac{(b+b^{-1})(q^{d+1}+q^{-d-1})+(c+c^{-1})(a+a^{-1})}{q+q^{-1}},$$

$$\alpha_c = \frac{(c+c^{-1})(q^{d+1}+q^{-d-1})+(a+a^{-1})(b+b^{-1})}{q+q^{-1}}.$$

Notation

By construction

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \alpha_a I,$$
$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \alpha_b I,$$
$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \alpha_c I.$$

The primitive idempotents

For $0 \leq i \leq d$ let $E_i \in \mathrm{End}(V)$ denote the projection from V onto the eigenspace of A for the eigenvalue θ_i.

We have

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq d),$$

$$I = \sum_{i=0}^{d} E_i, \qquad A = \sum_{i=0}^{d} \theta_i E_i.$$

Moreover $\{E_i\}_{i=0}^{d}$ is a basis for the subalgebra of $\mathrm{End}(V)$ generated by A.

We call $\{E_i\}_{i=0}^{d}$ the **primitive idempotents** for A.

Let $\{E'_i\}_{i=0}^{d}$ (resp. $\{E''_i\}_{i=0}^{d}$) denote the primitive idempotents for B (resp. C).

The maps W, W', W''

Define

$$W = \sum_{i=0}^{d} (-1)^i a^{-i} q^{i(d-i)} E_i,$$

$$W' = \sum_{i=0}^{d} (-1)^i b^{-i} q^{i(d-i)} E'_i,$$

$$W'' = \sum_{i=0}^{d} (-1)^i c^{-i} q^{i(d-i)} E''_i.$$

The maps W, W', W'', cont.

Each of W, W', W'' is invertible. Moreover

$$W^{-1} = \sum_{i=0}^{d} (-1)^i a^i q^{-i(d-i)} E_i,$$

$$(W')^{-1} = \sum_{i=0}^{d} (-1)^i b^i q^{-i(d-i)} E'_i,$$

$$(W'')^{-1} = \sum_{i=0}^{d} (-1)^i c^i q^{-i(d-i)} E''_i.$$

The maps W, W', W'', cont.

We have

$$WA = AW, \qquad W'B = BW', \qquad W''C = CW''.$$

The maps W, W', W'', cont.

We are going to describe the W, W', W'' in detail.

In order to motivate our results, we first consider the case $a = b = c$.

Theorem (Curtin 2006)

Assume that $a = b = c$. Then the following (i)–(iv) hold.

(i) *The Leonard triple A, B, C is modular.*

(ii) *We have*

$$W^{-1}BW = C, \qquad (W')^{-1}CW' = A, \qquad (W'')^{-1}AW'' = B.$$

(iii) $W'W = W''W' = WW''$.

(iv) *Denote the above common value by P. Then*

$$P^{-1}AP = B, \qquad P^{-1}BP = C, \qquad P^{-1}CP = A$$

and P^3 is a scalar multiple of the identity.

The case of general a, b, c

From now on, we drop the assumption $a = b = c$.

For $X \in \mathrm{End}(V)$ let $\langle X \rangle$ denote the subalgebra of $\mathrm{End}(V)$ generated by X.

Theorem (Terwilliger 2016)

We have

(i) $W^{-1}BW - C \in \langle A \rangle$;

(ii) $(W')^{-1}CW' - A \in \langle B \rangle$;

(iii) $(W'')^{-1}AW'' - B \in \langle C \rangle$.

The elements $\overline{A}$, $\overline{B}$, $\overline{C}$

Define

$$\begin{aligned}
\overline{A} &= W^{-1}BW - C, \\
\overline{B} &= (W')^{-1}CW' - A, \\
\overline{C} &= (W'')^{-1}AW'' - B.
\end{aligned}$$

So

$$\overline{A} \in \langle A \rangle, \qquad \overline{B} \in \langle B \rangle, \qquad \overline{C} \in \langle C \rangle.$$

The elements $\overline{A}$, $\overline{B}$, $\overline{C}$, cont.

Theorem (Terwilliger 2016)

We have

$$\overline{A}\left(I - \frac{A}{q+q^{-1}}\right) = (\alpha_b - \alpha_c)I = \left(I - \frac{A}{q+q^{-1}}\right)\overline{A};$$

$$\overline{B}\left(I - \frac{B}{q+q^{-1}}\right) = (\alpha_c - \alpha_a)I = \left(I - \frac{B}{q+q^{-1}}\right)\overline{B};$$

$$\overline{C}\left(I - \frac{C}{q+q^{-1}}\right) = (\alpha_a - \alpha_b)I = \left(I - \frac{C}{q+q^{-1}}\right)\overline{C}.$$

The elements $\overline{A}$, $\overline{B}$, $\overline{C}$, cont.

Consider the top equations in the previous slide.

As we seek to describe $\overline{A}$, it is tempting to invert the element $I - A/(q + q^{-1})$.

However this element might not be invertible.

We now investigate this possibility.

The invertibility of $\overline{A}$

Lemma

The following are equivalent:

(i) $I - \frac{A}{q+q^{-1}}$ *is invertible;*

(ii) $a \neq q^{d+1}$ *and* $a \neq q^{-d-1}$.

A description of $\overline{A}$

Theorem (Terwilliger 2016)

The element $\overline{A}$ is described as follows.

(i) *Assume $a = q^{d+1}$. Then*

$$\overline{A} = (b-c)(b-c^{-1})b^{-1}[d+1]_q E_0.$$

(ii) *Assume $a = q^{-d-1}$. Then*

$$\overline{A} = (b-c)(b-c^{-1})b^{-1}[d+1]_q E_d.$$

(iii) *Assume $a \neq q^{d+1}$ and $a \neq q^{-d-1}$. Then*

$$\overline{A} = (\alpha_b - \alpha_c)\left(I - \frac{A}{q+q^{-1}}\right)^{-1}.$$

Another description of $\overline{A}$

Theorem (Terwilliger 2016)

The element $\overline{A}$ is described as follows.

(i) *Assume $a = q^{d+1}$. Then $\overline{A}$ is equal to*

$$(b-c)(b-c^{-1})b^{-1}[d+1]_q \times \sum_{i=0}^{d} \frac{(A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{i-1} I)}{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_i)}.$$

(ii) *Assume $a = q^{-d-1}$. Then $\overline{A}$ is equal to*

$$(b-c)(b-c^{-1})b^{-1}[d+1]_q \times \frac{(A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{d-1} I)}{(\theta_d-\theta_0)(\theta_d-\theta_1)\cdots(\theta_d-\theta_{d-1})}.$$

Theorem

(Continued..)

(iii) *Assume* $a \neq q^{d+1}$ *and* $a \neq q^{-d-1}$. *Then* $\overline{A}$ *is equal to*

$$\frac{(a - q^{-d-1})(b - c)(b - c^{-1})b^{-1}q^d}{a - q^{d-1}}$$

times

$$\sum_{i=0}^{d} \frac{(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q + q^{-1} - \theta_1)(q + q^{-1} - \theta_2) \cdots (q + q^{-1} - \theta_i)}.$$

Some results about conjugation

Next we work out what happens if one of A, B, C is conjugated by one of $W^{\pm 1}$, $(W')^{\pm 1}$, $(W'')^{\pm 1}$.

We start with a result about W; similar results hold for W' and W''.

Theorem (Terwilliger 2016)

We have

$$WBW^{-1} - W^{-1}BW = \frac{AB - BA}{q - q^{-1}},$$

$$WCW^{-1} - W^{-1}CW = \frac{AC - CA}{q - q^{-1}}.$$

Some results on conjugation, cont.

Theorem (Terwilliger 2016)

We have

X	A	B	C
$W^{-1}XW$	A	$C+\overline{A}$	$B+\frac{CA-AC}{q-q^{-1}}-\overline{A}$
WXW^{-1}	A	$C+\frac{AB-BA}{q-q^{-1}}+\overline{A}$	$B-\overline{A}$
$(W')^{-1}XW'$	$C+\frac{AB-BA}{q-q^{-1}}-\overline{B}$	B	$A+\overline{B}$
$W'X(W')^{-1}$	$C-\overline{B}$	B	$A+\frac{BC-CB}{q-q^{-1}}+\overline{B}$
$(W'')^{-1}XW''$	$B+\overline{C}$	$A+\frac{BC-CB}{q-q^{-1}}-\overline{C}$	C
$W''X(W'')^{-1}$	$B+\frac{CA-AC}{q-q^{-1}}+\overline{C}$	$A-\overline{C}$	C

The elements W^2, $(W')^2$, $(W'')^2$.

We now consider W^2, $(W')^2$, $(W'')^2$.

As we will see, these elements are nice.

Lemma

We have

$$W^2 = \sum_{i=0}^{d} a^{-2i} q^{2i(d-i)} E_i,$$

$$(W')^2 = \sum_{i=0}^{d} b^{-2i} q^{2i(d-i)} E'_i,$$

$$(W'')^2 = \sum_{i=0}^{d} c^{-2i} q^{2i(d-i)} E''_i.$$

The elements W^2, $(W')^2$, $(W'')^2$, cont.

Lemma

We have

$$W^{-2} = \sum_{i=0}^{d} a^{2i} q^{-2i(d-i)} E_i,$$

$$(W')^{-2} = \sum_{i=0}^{d} b^{2i} q^{-2i(d-i)} E'_i,$$

$$(W'')^{-2} = \sum_{i=0}^{d} c^{2i} q^{-2i(d-i)} E''_i.$$

More results on conjugation

We next work out what happens if one of A, B, C is conjugated by one of $W^{\pm 2}$, $(W')^{\pm 2}$, $(W'')^{\pm 2}$.

Theorem (Terwilliger 2016)

We have

X	A	B	C
$W^{-2}XW^2$	A	$B+\frac{[C,A]}{q-q^{-1}}$	$C-\frac{[A,B]}{q-q^{-1}}+\frac{[A,[A,C]]}{(q-q^{-1})^2}$
W^2XW^{-2}	A	$B-\frac{[C,A]}{q-q^{-1}}+\frac{[A,[A,B]]}{(q-q^{-1})^2}$	$C+\frac{[A,B]}{q-q^{-1}}$
$(W')^{-2}X(W')^2$	$A-\frac{[B,C]}{q-q^{-1}}+\frac{[B,[B,A]]}{(q-q^{-1})^2}$	B	$C+\frac{[A,B]}{q-q^{-1}}$
$(W')^2X(W')^{-2}$	$A+\frac{[B,C]}{q-q^{-1}}$	B	$C-\frac{[A,B]}{q-q^{-1}}+\frac{[B,[B,C]]}{(q-q^{-1})^2}$
$(W'')^{-2}X(W'')^2$	$A+\frac{[B,C]}{q-q^{-1}}$	$B-\frac{[C,A]}{q-q^{-1}}+\frac{[C,[C,B]]}{(q-q^{-1})^2}$	C
$(W'')^2X(W'')^{-2}$	$A-\frac{[B,C]}{q-q^{-1}}+\frac{[C,[C,A]]}{(q-q^{-1})^2}$	$B+\frac{[C,A]}{q-q^{-1}}$	C

Note that $\overline{A}$, $\overline{B}$, $\overline{C}$ do not appear in the above table.

The elements $W^{\pm 1}$, $W^{\pm 2}$ as a polynomial in A

By construction $W^{\pm 1}$, $W^{\pm 2}$ are contained in the subalgebra $\langle A \rangle$ and are therefore polynomials in A.

Our next goal is to display these polynomials.

Some notation

We recall some notation. For $x, t \in \mathbb{F}$,

$$(x;t)_n = (1-x)(1-xt)\cdots(1-xt^{n-1}) \qquad\qquad n = 0,1,2,\ldots$$

We interpret $(x;t)_0 = 1$.

Theorem (Terwilliger 2016)

We have

$$W = \sum_{i=0}^{d} \frac{(-1)^i q^{i^2} (A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{i-1} I)}{(q^2;q^2)_i (aq^{1-d};q^2)_i},$$

$$W^{-1} = \sum_{i=0}^{d} \frac{(-1)^i a^i q^{i(i-d+1)} (A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{i-1} I)}{(q^2;q^2)_i (aq^{1-d};q^2)_i}.$$

The elements $W^{\pm 2}$ as a polynomial in A

Theorem (Terwilliger 2016)

We have

$$W^2 = \sum_{i=0}^{d} \frac{a^{-i} q^{id} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i},$$

$$W^{-2} = \sum_{i=0}^{d} \frac{(-1)^i a^i q^{i(i-d+1)} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i}.$$

A product

The elements W^2, $(W')^2$, $(W'')^2$ are related as follows.

Theorem (Terwilliger 2016)

We have

$$(W'')^2(W')^2W^2 = (abc)^{-d}q^{d(d-1)}I. \quad (1)$$

The elements $W'W$, $W''W'$, WW''

Consider the three elements

$$W'W, \qquad W''W', \qquad WW''.$$

Our next goal is to show that these elements mutually commute, and their product is a scalar multiple of the identity.

Our strategy is to bring in the element $A + B + C$.

The element $A + B + C$

Lemma

Each of the elements

$$W'W, \qquad W''W', \qquad WW''$$

commutes with $A + B + C$.

Recall that $\langle A + B + C \rangle$ is the $\mathbb{F}$-subalgebra of $\mathrm{End}(V)$ generated by $A + B + C$.

Lemma

There exists $v \in V$ such that $\langle A + B + C \rangle v = V$.

Lemma

The subalgebra $\langle A + B + C \rangle$ contains every element of $\mathrm{End}(V)$ that commutes with $A + B + C$. In particular $\langle A + B + C \rangle$ contains each of the elements

$$W'W, \qquad W''W', \qquad WW''.$$

The elements $W'W$, $W''W'$, WW''

Theorem (Terwilliger 2016)

The elements

$$W'W, \qquad W''W', \qquad WW''$$

mutually commute.

Theorem (Terwilliger 2016)

The product of the elements

$$W'W, \qquad W''W', \qquad WW''$$

is equal to $(abc)^{-d}q^{d(d-1)}I$.

Proof.

Observe

$$\begin{aligned}(W'W)(W''W')(WW'') &= (WW'')(W''W')(W'W) \\ &= W(W'')^2(W')^2W^2W^{-1} \\ &= (abc)^{-d}q^{d(d-1)}WW^{-1} \\ &= (abc)^{-d}q^{d(d-1)}I.\end{aligned}$$

□

Summary

This talk was about a Leonard triple A, B, C of q-Racah type.

We described this triple, using three invertible linear maps called W, W', W''.

We saw that

- A commutes with W and $W^{-1}BW - C$;
- B commutes with W' and $(W')^{-1}CW' - A$;
- C commutes with W'' and $(W'')^{-1}AW'' - B$.

Moreover the three elements $W'W, W''W', WW''$ mutually commute, and their product is a scalar multiple of the identity.

Thank you for your attention!

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