The subconstituent algebra of a graph, the Q-polynomial property, and tridiagonal pairs of linear transformations

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Overview

This survey talk has two parts. In Part I, we review the **subconstituent algebra** T of a graph.

We will discuss the Q-**polynomial** assumption, under which T is well behaved.

Motivated by the first part, in Part II we discuss a linear-algebraic object called a **tridiagonal pair**.

A tridiagonal pair consists of two diagonalizable linear transformations on a nonzero finite-dimensional vector space, that each act in a (block)-tridiagonal fashion on the eigenspaces of the other one.

We will discuss the classification of tridiagonal pairs, and describe in detail a special case called a **Leonard pair**.

Contents

Throughout this talk, all vector spaces and algebras are understood to be over $\mathbb{C}$.

All algebras are understood to be associative and have a multiplicative identity.

Let X denote a nonempty finite set.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ consists of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

The vector space $V = \mathbb{C}^X$ consists of the column vectors with rows indexed by X and all entries in $\mathbb{C}$.

$\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

Endow V with a Hermitian inner product

$$\langle u, v \rangle = u^t \overline{v} \qquad (u, v \in V).$$

For each $x \in X$ let $\hat{x}$ denote the vector in V that has a 1 in coordinate x and 0 in all other coordinates.

The vectors $\{\hat{x} | x \in X\}$ form an orthonormal basis for V.

The graph Γ

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, adjacency relation $\mathcal{R}$, and path-length distance function ∂.

For an integer $i \geq 0$ and $x \in X$, define the set

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

The graphs of interest

Our main case of interest is when Γ is "highly regular" in a certain way.

A good example to keep in mind is the D-**dimensional hypercube**, also called the **binary Hamming graph** $H(D, 2)$.

Note that $H(2, 2)$ is a 4-cycle; this will be used as a running example.

The adjacency matrix

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the $(0,1)$-adjacency matrix of Γ.

For $x \in X$,

$$A\hat{x} = \sum_{y \in \Gamma(x)} \hat{y}.$$

Example

For $H(2,2)$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The adjacency algebra M

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.

M is called the **adjacency algebra** of Γ.

M is commutative and finite-dimensional.

Example

For $H(2,2)$, M has a basis

$$I, A, J$$

where the matrix J has every entry 1.

The matrix A is real and symmetric, so A is diagonalizable and its eigenvalues are real.

Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the distinct eigenvalues of A.

For $0 \leq i \leq d$ let the matrix $E_i \in \mathrm{Mat}_X(\mathbb{C})$ act as the identity on the θ_i-eigenspace of A, and as zero on every other eigenspace of A.

We call E_i the **primitive idempotent** of A (or Γ) associated with θ_i.

The primitive idempotents, cont.

We have

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq d),$$

$$I = \sum_{i=0}^{d} E_i,$$

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

The matrices $\{E_i\}_{i=0}^{d}$ form a basis for M.

Example

For $H(2,2)$ we have $\theta_0 = 2$, $\theta_1 = 0$, $\theta_2 = -2$. Moreover

$$E_0 = 1/4J,$$

$$E_1 = 1/4 \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix},$$

$$E_2 = 1/4 \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The eigenspaces of A

The vector space V decomposes as

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum)}.$$

For $0 \leq i \leq d$ the subspace $E_i V$ is the θ_i-eigenspace of A.

Fix a vertex x of Γ

Until further notice

$$\textbf{fix } x \in X.$$

We call x the **base vertex**.

Define $D = D(x)$ by

$$D = \max\{\partial(x, y) \mid y \in X\}.$$

We call D the **diameter of Γ with respect to x**.

The dual primitive idempotents

For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i. \end{cases} \qquad (y \in X).$$

We call E_i^* the ith **dual primitive idempotent of Γ with respect to x**.

The dual primitive idempotents, cont.

Example

For $H(2,2)$,

$$E_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_1^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual adjacency algebra.

The $\{E_i^*\}_{i=0}^D$ satisfy

$$E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \leq i, j \leq D),$$

$$I = \sum_{i=0}^{D} E_i^*.$$

The matrices $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ denoted $M^* = M^*(x)$.

We call M^* the **dual adjacency algebra of Γ with respect to x.**

The subconstituents of Γ with respect to x

The vector space V decomposes as

$$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(orthogonal direct sum)}.$$

The above summands are the common eigenspaces for M^*.

For $0 \leq i \leq D$,

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in \Gamma_i(x)\}.$$

We call $E_i^* V$ the i**th subconstituent of Γ with respect to** x.

So far, we defined the adjacency algebra M and the dual adjacency algebra M^*. We now combine M and M^* to get a larger algebra.

Definition (Ter 92)

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the **subconstituent algebra of Γ with respect to** x.

T is finite-dimensional.

T is noncommutative in general.

The algebra T is semi-simple

By a T-**module** we mean a subspace $W \subseteq V$ such that $TW \subseteq W$.

The algebra T is closed under the conjugate-transpose map.

So for each T-module W, its orthogonal complement $W^\perp$ is a T-module.

Therefore, the T-module V is an orthogonal direct sum of irreducible T-modules.

This means that T is semi-simple.

Since T is semi-simple, by the Wedderburn theory T is isomorphic to a direct sum of matrix algebras.

Example

For $H(2,2)$,

$$T \simeq \mathrm{Mat}_3(\mathbb{C}) \oplus \mathbb{C}.$$

Moreover $\dim(T) = 10$.

The algebra T is semisimple, cont.

Example (Junie Go 2002)

For $H(D, 2)$,

$$T \simeq \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-3}(\mathbb{C}) \oplus \cdots$$

Moreover $\dim(T) = \binom{D+3}{3}$.

A research problem

The above comments motivate the following research problem.

Problem

(i) How does the Wedderburn decomposition of T reflect the combinatorial properties of Γ?

(ii) For which graphs Γ is this decomposition particularly nice?

The dual adjacency matrix

We now describe a family of graphs for which the Wedderburn decomposition of T is nice.

These graphs possess a certain matrix called a **dual adjacency matrix**.

We will define this type of matrix shortly.

By the triangle inequality

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

This is reformulated as follows.

Lemma

For $0 \leq i, j \leq D$,

$$E_i^* A E_j^* = 0 \quad \textit{if } |i - j] > 1.$$

The dual adjacency matrix

Definition

Referring to the graph Γ, consider a matrix $A^* \in \mathrm{Mat}_X(\mathbb{C})$ that satisfies the following conditions:

(i) A^* generates M^*;

(ii) for $0 \le i, j \le d$,

$$E_i A^* E_j = 0 \quad \text{if } |i - j] > 1.$$

We call A^* a **dual adjacency matrix of** Γ (with respect to x and the given ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents).

A dual adjacency matrix A^* is **diagonal**.

The dual adjacency matrix, cont.

For a dual adjacency matrix A^* the eigenspaces are

$$E_i^* V \qquad (0 \leq i \leq D).$$

The matrix A^* acts on the eigenspaces of A as follows:

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

Example

$H(2,2)$ has a dual adjacency matrix

$$A^* = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Example (Junie Go 2002)

The hypercube $H(D,2)$ has a dual adjacency matrix

$$A^* = \sum_{i=0}^{D} (D-2i) E_i^*.$$

How A, A^* are related

We now consider how the adjacency matrix A and any dual adjacency matrix A^* are related.

Example (Junie Go 2002)

For the hypercube $H(D,2)$ we have

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A.$$

More generally we have the following.

Theorem (Ito+Tanabe+T, 2001)

For the graph Γ *with adjacency matrix* A *and dual adjacency matrix* A^**, there exist complex scalars* $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ *such that*

$$\begin{aligned} & A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ &= \quad \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} & A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ &= \quad \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

These equations are called the **tridiagonal relations**.

The Q-polynomial property

Definition

The graph Γ is called **Q-polynomial** (with respect to x and the given ordering of the primitive idempotents) whenever Γ has a dual adjacency matrix with respect to x and $\{E_i\}_{i=0}^d$.

For example, the hypercube $H(D, 2)$ is Q-polynomial with respect to every vertex.

We now give some more examples.

Q-polynomial examples

The following graphs are Q-polynomial with respect to every vertex:

- strongly-regular graph
- cycle
- Hamming graph
- Johnson graph
- Grassmann graph
- Dual polar space graph
- Bilinear forms graph
- Alternating forms graph
- Hermitian forms graph
- Quadratic forms graph

See the book **Distance-Regular Graphs** by Brouwer, Cohen, Neumaier.

How A and A^* are related

To summarize so far, for our graph Γ the adjacency matrix A and any dual adjacency matrix A^* generate T. Moreover A, A^* act on each other's eigenspaces in the following tridiagonal way:

$$AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \qquad (0 \le i \le D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$;

$$A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \le i \le d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

To investigate this situation, we reformulate it using the language of linear algebra.

We now define a linear-algebraic object called a **tridiagonal pair**.

Going forward, V will denote a nonzero vector space with finite dimension.

We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$.

Definition of a tridiagonal pair

We call A, A^* a **tridiagonal pair** on V whenever:

- Each of A, A^* is diagonalizable on V.
- There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

  where $V_{-1} = 0$ and $V_{d+1} = 0$.
- There exists an ordering $\{V_i^*\}_{i=0}^D$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq D),$$

  where $V_{-1}^* = 0$ and $V_{D+1}^* = 0$.
- There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^* W \subseteq W$ and $W \neq 0$ and $W \neq V$.

The diameter

Referring to our definition of a tridigonal pair,

it turns out that $d = D$; we call this common value the **diameter** of the pair.

Each irreducible T-module gives a tridiagonal pair

Briefly returning to the graph Γ, the adjacency matrix and any dual adjacency matrix act on each irreducible T-module as a tridiagonal pair.

This motivates us to understand tridiagonal pairs.

The classification of tridiagonal pairs

After 10 years of work and several dozen papers, my collaborators Tatsuro Ito, Kazumasa Nomura and I classified up to isomorphism the tridiagonal pairs; see

T. Ito, K. Nomura, P. Terwilliger The classification of the sharp tridiagonal pairs. Linear Algebra Appl. 435 (2011) 1857–1884.

The classification is a bit involved, so I will skip over it.

Instead, I will give the general idea by discussing the following special case.

Definition

A **Leonard pair** on V is a tridiagonal pair A, A^* on V such that for each of A, A^* every eigenspace has dimension one.

The name Leonard pair is in honor of Douglas Leonard (Auburn U.) whose 1982 theorem about the q-Racah polynomials motivated this topic.

When working with a Leonard pair, it is helpful to consider a closely related object called a **Leonard system**.

We will define a Leonard system over the next few slides.

Standard orderings

Let A, A^* denote a Leonard pair on V with diameter d.

An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

In this case, the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard.

A similar discussion applies to A^*.

Primitive idempotents

Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** acts as the identity on that eigenspace, and as zero on every other eigenspace.

Definition

By a **Leonard system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies the following:

- A, A^* is a Leonard pair on V;
- $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A;
- $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

Until further notice, we fix a Leonard system Φ on V as above.

The relatives of Φ

Using Φ we get some more Leonard systems on V:

$$\Phi^* = (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),$$
$$\Phi^\downarrow = (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),$$
$$\Phi^\Downarrow = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

We call these Leonard systems the **relatives of** Φ.

The eigenvalues and dual eigenvalues

Definition

For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E^*_i).

By construction

$$\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad \text{if } i \neq j \quad (0 \leq i, j \leq d).$$

We call the sequence $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

The Φ-standard basis

Next, we use Φ to obtain an attractive basis for V.

Lemma

Pick $0 \neq \xi \in E_0 V$.

(i) *For* $0 \leq i \leq d$ *the vector* $E_i^* \xi$ *is nonzero and hence a basis for* $E_i^* V$.

(ii) *The vectors* $\{E_i^* \xi\}_{i=0}^d$ *form a basis for* V.

(iii) $\xi = \sum_{i=0}^d E_i^* \xi$.

Definition

For $0 \neq \xi \in E_0 V$ the basis $\{E_i^* \xi\}_{i=0}^d$ of V is called **Φ-standard**.

The intersection numbers of Φ

Lemma

With respect to a Φ-standard basis for V the matrices representing A and A^* are

$$A: \quad \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix},$$

$$A^*: \quad \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d),$$

where $b_{i-1}c_i \neq 0$ for $1 \leq i \leq d$.

The intersection numbers of Φ, cont.

Definition

We call $\{a_i\}_{i=0}^{d}$, $\{b_i\}_{i=0}^{d-1}$, $\{c_i\}_{i=1}^{d}$ the **intersection numbers** of Φ.

Lemma

We have

$$c_i + a_i + b_i = \theta_0 \qquad (0 \leq i \leq d),$$

where $c_0 = 0$ *and* $b_d = 0$.

Next, we interchange the roles of A, A^*.

The dual intersection numbers of Φ

Lemma

With respect to a Φ^*-standard basis for V, the matrices representing A and A^* are

$$A: \quad \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_d),$$

$$A^*: \quad \begin{pmatrix} a_0^* & b_0^* & & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & & \\ & c_2^* & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1}^* \\ \mathbf{0} & & & & c_d^* & a_d^* \end{pmatrix}$$

where $b_{i-1}^* c_i^* \neq 0$ for $1 \leq i \leq d$.

Definition

We call $\{a_i^*\}_{i=0}^{d}$, $\{b_i^*\}_{i=0}^{d-1}$, $\{c_i^*\}_{i=1}^{d}$ the **dual intersection numbers** of Φ.

Lemma

We have

$$c_i^* + a_i^* + b_i^* = \theta_0^* \qquad (0 \le i \le d),$$

where $c_0^* = 0$ *and* $b_d^* = 0$.

Some polynomials

Next, we bring in some polynomials.

Let λ denote an indeterminate. Let the algebra $\mathbb{C}[\lambda]$ consist of the polynomials in λ that have all coefficients in $\mathbb{C}$.

Definition

We define some polynomials $\{u_i\}_{i=0}^d$ in $\mathbb{C}[\lambda]$ such that

$$u_0 = 1,$$
$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (0 \leq i \leq d-1),$$

where $u_{-1} = 0$.

The polynomial u_i has degree i for $0 \leq i \leq d$.

More polynomials

Definition

We define some polynomials $\{u_i^*\}_{i=0}^d$ in $\mathbb{C}[\lambda]$ such that

$$u_0^* = 1,$$
$$\lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \qquad (0 \le i \le d-1),$$

where $u_{-1}^* = 0$.

The polynomial u_i^* has degree i for $0 \le i \le d$.

The significance of the polynomials

The polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ have the following significance.

Lemma

For $0 \le i \le d$ we have

$$u_i(A)E_0^* V = E_i^* V, \qquad u_i^*(A^*)E_0 V = E_i V.$$

The polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ are related to each other in the following way.

Lemma

For $0 \leq i, j \leq d$,

$$u_i(\theta_j) = u_j^*(\theta_i^*).$$

This relationship is called **Askey-Wilson duality**.

The first and second split sequence

Shortly, we will give explicit formula for the intersection numbers and dual intersection numbers of Φ.

To obtain these formula, it is convenient to bring in some additional scalar parameters.

These parameters form two sequences, called the **first split sequence** and **second split sequence**.

We will define these sequences on the next slides.

Lemma

There exist nonzero complex scalars $\{\varphi_i\}_{i=1}^d$ *and a basis for* V *with respect to which*

$$
A: \quad \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix},
$$

$$
A^*: \quad \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.
$$

The first split sequence, cont.

Definition

The sequence $\{\varphi_i\}_{i=1}^{d}$ is called the **first split sequence** of Φ.

Next, we apply the previous lemma and definition to the Leonard system $\Phi^{\Downarrow}$.

The second split sequence

Lemma

There exist nonzero complex scalars $\{\phi_i\}_{i=1}^d$ *and a basis for* V *with respect to which*

$$A: \quad \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix},$$

$$A^*: \quad \begin{pmatrix} \theta^*_0 & \phi_1 & & & \mathbf{0} \\ & \theta^*_1 & \phi_2 & & \\ & & \theta^*_2 & \cdot & \\ & & & \cdot & \cdot \\ & & & \cdot & \phi_d \\ \mathbf{0} & & & & \theta^*_d \end{pmatrix}.$$

The second split sequence, cont.

Definition

The sequence $\{\phi_i\}_{i=1}^{d}$ is called the **second split sequence** of Φ.

Definition

By the **parameter array** of Φ we mean the sequence

$$\big(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\big)$$

where

- $\{\theta_i\}_{i=0}^d$ is the eigenvalue sequence of Φ;
- $\{\theta^*_i\}_{i=0}^d$ is the dual eigenvalue sequence of Φ;
- $\{\varphi_i\}_{i=1}^d$ is the first split sequence of Φ;
- $\{\phi_i\}_{i=1}^d$ is the second split sequence of Φ.

The significance of the parameter array

The parameter array of Φ has the following significance.

Lemma

The Leonard system Φ is uniquely determined up to isomorphism by its parameter array.

The significance of the parameter array, cont.

The previous lemma suggests that we can write the intersection numbers, the dual intersection numbers, and the polynomials $\{u_i\}_{i=0}^d$, $\{u_i^*\}_{i=0}^d$ in terms of the parameter array.

We will do this in the upcoming slides.

The intersection numbers and dual intersection numbers in terms of the parameter array

Next, for Φ we give the intersection numbers and dual intersection numbers in terms of the parameter array.

For $0 \leq i \leq d-1$,

$$b_i = \varphi_{i+1} \frac{(\theta^*_i - \theta^*_0)(\theta^*_i - \theta^*_1) \cdots (\theta^*_i - \theta^*_{i-1})}{(\theta^*_{i+1} - \theta^*_0)(\theta^*_{i+1} - \theta^*_1) \cdots (\theta^*_{i+1} - \theta^*_i)},$$

$$b^*_i = \varphi_{i+1} \frac{(\theta_i - \theta_0)(\theta_i - \theta_1) \cdots (\theta_i - \theta_{i-1})}{(\theta_{i+1} - \theta_0)(\theta_{i+1} - \theta_1) \cdots (\theta_{i+1} - \theta_i)}.$$

The intersection numbers and dual intersection numbers in terms of the parameter array, cont.

For $1 \le i \le d$,

$$c_i = \phi_i \frac{(\theta^*_i - \theta^*_d)(\theta^*_i - \theta^*_{d-1}) \cdots (\theta^*_i - \theta^*_{i+1})}{(\theta^*_{i-1} - \theta^*_d)(\theta^*_{i-1} - \theta^*_{d-1}) \cdots (\theta^*_{i-1} - \theta^*_i)},$$

$$c^*_i = \phi_{d-i+1} \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{i+1})}{(\theta_{i-1} - \theta_d)(\theta_{i-1} - \theta_{d-1}) \cdots (\theta_{i-1} - \theta_i)}.$$

The polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ in terms of the parameter array

Next, for Φ we express the polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ in terms of the parameter array.

For $0 \le i \le d$,

$$u_i(\lambda) = \sum_{n=0}^{i} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*)(\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n},$$

$$u_i^*(\lambda) = \sum_{n=0}^{i} \frac{(\theta_i - \theta_0)(\theta_i - \theta_1)\cdots(\theta_i - \theta_{n-1})(\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{n-1}^*)}{\varphi_1\varphi_2\cdots\varphi_n}.$$

The above forms imply the Askey-Wilson duality that we encountered earlier.

For $0 \leq i, j \leq d$ the common value of $u_i(\theta_j) = u^*_j(\theta^*_i)$ is

$$\sum_{n=0}^{\min(i,j)} \frac{(\theta^*_i - \theta^*_0)(\theta^*_i - \theta^*_1)\cdots(\theta^*_i - \theta^*_{n-1})(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}.$$

We are now ready to give the classification theorem for Leonard systems.

Theorem (Ter 2001)

Given complex scalars $\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$ $(*)$, *there exists a Leonard system* Φ *with parameter array* $(*)$ *iff*

- $\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad$ *if* $\quad i \neq j, \qquad (0 \leq i,j \leq d);$
- $\varphi_i \neq 0, \quad \phi_i \neq 0 \qquad (1 \leq i \leq d);$
- $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d);$
- $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d);$
- *the scalars*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i *for* $2 \leq i \leq d-1$.

The classification of Leonard systems, cont.

Theorem (Ter 2001)

...(Continued)... Moreover, if Φ exists then Φ is unique up to isomorphism of Leonard systems.

The solutions $(*)$ to the classification theorem can be expressed in parametric form.

To illustrate, we give the most general solution.

The q-Racah polynomials

Referring to the classification theorem, the most general solution is

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

for $0 \le i \le d$ and

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i),$$
$$\phi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^*$$

for $1 \le i \le d$, where $r_1r_2 = ss^*q^{d+1}$.

For this solution,

$$b_0 = \frac{h(1-q^{-d})(1-r_1q)(1-r_2q)}{1-s^*q^2},$$

$$b_i = \frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} \quad (1 \leq i \leq d-1),$$

$$c_i = \frac{h(1-q^{i})(1-s^*q^{i+d+1})(r_1-s^*q^{i})(r_2-s^*q^{i})}{s^*q^d(1-s^*q^{2i})(1-s^*q^{2i+1})} \quad (1 \leq i \leq d-1),$$

$$c_d = \frac{h(1-q^{d})(r_1-s^*q^{d})(r_2-s^*q^{d})}{s^*q^d(1-s^*q^{2d})}.$$

To get $\{b_i^*\}_{i=0}^{d-1}$ and $\{c_i^*\}_{i=1}^{d}$, in the above formulas exchange $h \leftrightarrow h^*$, $s \leftrightarrow s^*$ and preserve r_1, r_2, q.

For the above solution, the polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ look as follows.

For $0 \le i, j \le d$ the common value of $u_i(\theta_j) = u_j^*(\theta_i^*)$ is

$$\sum_{n=0}^{\min(i,j)} \frac{(q^{-i};q)_n (s^* q^{i+1};q)_n (q^{-j};q)_n (s q^{j+1};q)_n q^n}{(r_1 q;q)_n (r_2 q;q)_n (q^{-d};q)_n (q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

for $n = 0, 1, 2 \ldots$

The above sum is a **basic hypergeometric series**; the notation is

$$
{}_4\phi_3\left(\begin{matrix} q^{-i}, s^* q^{i+1}, \ q^{-j}, \ sq^{j+1} \\ r_1 q, \ \ r_2 q, \ \ q^{-d} \end{matrix} \,\middle|\, q, q\right).
$$

For this solution the polynomials $\{u_i\}_{i=0}^d$ and $\{u_i^*\}_{i=0}^d$ are in the q-**Racah class**.

For the classification of Leonard systems, altogether the solutions fall into 12 families. These families correspond to the following classes of orthogonal polynomials:

q-Racah,
q-Hahn,
dual q-Hahn,
q-Krawtchouk,
dual q-Krawtchouk,
quantum q-Krawtchouk,
affine q-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito.

The Askey scheme of orthogonal polynomials, cont.

The above polynomial classes form the **terminating branch of the Askey scheme** of orthogonal polynomials.

Roughly speaking, the Leonard systems are the same thing (in disguise) as the orthogonal polynomials from the terminating branch of the Askey scheme.

Conclusion

We have seen how Leonard pairs arise in connection with the subconstituent algebra of a graph.

In fact, Leonard pairs arise in many areas of mathematics and physics.

Indeed, Leonard pairs arise wherever the following polynomials are found:

q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, affine q-Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito.

Bibliography

P. Terwilliger:
An algebraic approach to the Askey scheme of orthogonal polynomials.
Orthogonal polynomials and special functions,
255–330, Lecture Notes in Math., 1883,
Springer, Berlin, 2006;
arXiv:math.QA/0408390.

P. Terwilliger:
Notes on the Leonard system classification.
Graphs and Combinatorics 37 (2021) 1687–1748;
arXiv:2003.09668.

THANK YOU FOR YOUR ATTENTION!