The universal Askey-Wilson algebra

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Overview

This talk concerns an algebra Δ called the **Universal Askey-Wilson algebra**.

As we will see, Δ is related to:

- Leonard pairs and Leonard triples of QRacah type
- Q-polynomial distance-regular graphs of QRacah type
- The modular group $\mathrm{PSL}_2(\mathbb{Z})$
- The equitable presentation of the quantum group $U_q(\mathfrak{sl}_2)$
- The double affine Hecke algebra of type $(C_1^\vee, C_1)$

Motivation: Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

The Definition of a Leonard Pair

We now define a Leonard pair. From now on $\mathbb{F}$ will denote a field.

Definition (Terwilliger 1999)

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a **Leonard pair** on V, we mean a pair of linear transformations $A : V \to V$ and $B : V \to V$ such that:

1. There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal.
2. There exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is irreducible tridiagonal.

Leonard pairs in summary

In summary, for a Leonard pair A, B

	A	B
basis 1	diagonal	irred. tridiagonal
basis 2	irred. tridiagonal	diagonal

Leonard pairs and orthogonal polynomials

The term **Leonard pair** is motivated by a 1982 theorem of **Doug Leonard** concerning the QRacah polynomials and some related polynomials in the Askey scheme.

For a detailed version of Leonard's theorem see the book

E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin Cummings. London 1984.

Here is an example of a Leonard pair.

Fix an integer $d \geq 3$.
Pick nonzero scalars a, b, c, q in $\mathbb{F}$ such that

(i) $q^{2i} \neq 1$ for $1 \leq i \leq d$;

(ii) Neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$;

(iii) None of abc, $a^{-1}bc$, $ab^{-1}c$, abc^{-1} is among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

Leonard pair example, cont.

Define

$$\begin{aligned} \theta_i &= aq^{2i-d} + a^{-1}q^{d-2i}, \\ \theta_i^* &= bq^{2i-d} + b^{-1}q^{d-2i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i = \; & a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1}) \\ & (q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$.

Leonard pair example, cont.

Define

$$
A = \begin{pmatrix}
\theta_0 & & & & & \mathbf{0} \\
1 & \theta_1 & & & & \\
& 1 & \theta_2 & & & \\
& & \cdot & \cdot & & \\
& & & \cdot & \cdot & \\
\mathbf{0} & & & & 1 & \theta_d
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
\theta_0^* & \varphi_1 & & & & \mathbf{0} \\
& \theta_1^* & \varphi_2 & & & \\
& & \theta_2^* & \cdot & & \\
& & & \cdot & \cdot & \\
& & & & \cdot & \varphi_d \\
\mathbf{0} & & & & & \theta_d^*
\end{pmatrix}
$$

Leonard pair example cont.

Then the pair A, B is a Leonard pair on the vector space $V = \mathbb{F}^{d+1}$.

A Leonard pair of this form is said to have **QRacah type**.

This is the most general type of Leonard pair.

Hau-wen Huang (former student of **Chih-wen Weng** in the Department of Applied Math, National Chiao Tung University, Taiwan) has proven the following beautiful theorem about the Leonard pairs of QRacah type.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

Theorem (Hau-wen Huang 2011)

Referring to the above Leonard pair A, B of QRacah type, there exists an element C such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}} I,$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}} I,$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}} I.$$

The above equations are called the $\mathbb{Z}_3$-**symmetric Askey-Wilson relations**.

Leonard triples

In the previous example the $\mathbb{Z}_3$-symmetry involving A, B, C suggests that we should consider Leonard triples along with Leonard pairs.

The notion of a Leonard triple is due to **Brian Curtin** and defined as follows.

Definition (Brian Curtin 2007)

By a **Leonard triple** on V we mean an ordered triple of linear transformations (A, B, C) in $\mathrm{End}(V)$ such that for each $\phi \in \{A, B, C\}$ there exists a basis for V with respect to which the matrix representing ϕ is diagonal and the matrices representing the other two linear transformations are irreducible tridiagonal.

Leonard triples in summary

In summary, for a Leonard triple A, B, C

	A	B	C
basis 1	diagonal	irred. tridiagonal	irred. tridiagonal
basis 2	irred. tridiagonal	diagonal	irred. tridiagonal
basis 3	irred. tridiagonal	irred. tridiagonal	diagonal

For a moment let us return to our Leonard pair A, B of QRacah type.

Consider the element C from the $\mathbb{Z}_3$-symmetric Askey-Wilson relations.

Huang has found necessary and sufficient conditions on C for the triple A, B, C to be a Leonard triple.

This is explained in the next two theorems.

Leonard pairs and Leonard triples, cont.

Theorem (Hau-wen Huang 2011)

The roots of the characteristic polynomial of C are $\{\theta_i^{\varepsilon}\}_{i=0}^{d}$, where

$$\theta_i^{\varepsilon} = cq^{2i-d} + c^{-1}q^{d-2i} \qquad (0 \leq i \leq d)$$

Leonard pairs and Leonard triples, cont.

Theorem (Hau-wen Huang 2011)

The following (i)–(iii) *are equivalent.*

(i) *The triple* A, B, C *is a Leonard triple;*

(ii) $\{\theta_i^{\varepsilon}\}_{i=0}^{d}$ *are mutually distinct;*

(iii) c^2 *is not among* $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.

The above Leonard triple is said to have **QRacah type**.

The Askey-Wilson algebra

In 1992 **Alexei Zhedanov** introduced the **Askey-Wilson algebra** AW=AW(3) and used it to describe the Askey-Wilson polynomials.

Essentially, AW is the algebra defined by three generators A, B, C subject to the $\mathbb{Z}_3$-symmetric Askey-Wilson relations (The original definition was somewhat different).

The algebra AW is defined using four parameters a, b, c, q.

We now define a central extension of AW, called the **universal Askey-Wilson algebra** and denoted Δ.

The algebra Δ involves just one parameter q.

The algebra Δ is defined as follows.

For the rest of the talk, q denotes a nonzero scalar in $\mathbb{F}$ such that $q^4 \neq 1$.

Definition (Ter 2011)

Define an $\mathbb{F}$-algebra $\Delta = \Delta_q$ by generators and relations in the following way. The generators are A, B, C. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ. We call Δ the **universal Askey-Wilson algebra**.

By constuction, each Askey-Wilson algebra AW is a homomorphic image of Δ.

By construction, each Leonard pair or triple of QRacah type can be viewed as a Δ-module.

Δ and Q-polynomial DRGs

We now briefly relate Δ to Q-polynomial distance-regular graphs.

Let Γ denote a distance-regular graph with diameter $D \geq 3$ and distance matrices $\{A_i\}_{i=0}^{D}$.

Assume Γ has a Q-polynomial ordering $\{E_i\}_{i=0}^{D}$ of its primitive idempotents.

Assume that the Q-polynomial structure has QRacah type; this means (in the notation of Bannai/Ito) type I with each of s, s^* nonzero.

Δ and Q-polynomial DRGs, cont.

Fix a vertex x of Γ. For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the dual distance matrix of Γ that corresponds to E_i and x.

Assume that each irreducible T-module is thin. Here $T = T(x)$ is the subconstituent algebra of Γ with respect to x, generated by A_1 and A_1^*.

Theorem (Arjana Zitnik, Ter, in preparation)

With the above assumptions and notation, there exists a surjective algebra homomorphism $\Delta \to T$ that sends the generator A to a linear combination of I, A_1 and the generator B to a linear combination of I, A_1^*.

Three central elements of Δ

We now describe Δ from a ring theoretic point of view.

Definition

Define elements α, β, γ of Δ such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}},$$
$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}},$$
$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}.$$

Note that each of α, β, γ is central in Δ.

A basis for Δ

Theorem (Ter 2011)

The following is a basis for the $\mathbb{F}$-vector space Δ:

$$A^i B^j C^k \alpha^r \beta^s \gamma^t \qquad\qquad i, j, k, r, s, t \in \mathbb{N}.$$

We proved this using the Bergman Diamond Lemma.

An action of $\mathrm{PSL}_2(\mathbb{Z})$ on Δ

Recall that the modular group $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators p, s and relations $p^3 = 1, s^2 = 1$.

Our next goal is to show that $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ as a group of automorphisms.

Strategy: identify two automorphisms of Δ that have orders 3 and 2.

By construction Δ has an automorphism that sends

$$A \mapsto B \mapsto C \mapsto A.$$

This automorphism has order 3.

To find an automorphism of Δ that has order 2, we use another presentation for Δ.

Alternate presentation for Δ

Theorem (Ter 2011)

The algebra Δ has a presentation by generators A, B, γ and relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = -(q^2 - q^{-2})^2(AB - BA),$$
$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = -(q^2 - q^{-2})^2(BA - AB),$$
$$A^2B^2 - B^2A^2 + (q^2 + q^{-2})(BABA - ABAB) = -(q - q^{-1})^2(AB - BA)\gamma,$$
$$\gamma A = A\gamma, \qquad \gamma B = B\gamma.$$

Here $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

The first two relations above are the **tridiagonal relations**.

An automorphism of Δ that has order 2

By the alternate presentation Δ has an automorphism that swaps A, B and fixes γ.

This automorphism has order 2.

An action of $\mathrm{PSL}_2(\mathbb{Z})$ on Δ

Theorem (Ter 2011)

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ as a group of automorphisms in the following way:

u	A	B	C	α	β	γ
$p(u)$	B	C	A	β	γ	α
$s(u)$	B	A	$C + \frac{AB-BA}{q-q^{-1}}$	β	α	γ

This action is faithful.

The Casimir element Ω of Δ

Shortly we will describe the center $Z(\Delta)$.

To do this we introduce a certain element $\Omega \in \Delta$ called the Casimir element.

Definition

Define

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma.$$

We call Ω the **Casimir element** of Δ.

The Casimir element Ω is central

Theorem (Ter 2011)

The Casimir element Ω is contained in $Z(\Delta)$.

Moreover Ω is fixed by everything in $\mathrm{PSL}_2(\mathbb{Z})$.

A basis for Δ that involves Ω

We are going to show that $Z(\Delta)$ is generated by $\Omega, \alpha, \beta, \gamma$ provided that q is not a root of unity.

To this end we display a basis for Δ that involves Ω.

Lemma (Ter 2011)

The following is a basis for the $\mathbb{F}$-vector space Δ:

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t \qquad i, j, k, \ell, r, s, t \in \mathbb{N} \qquad ijk = 0.$$

Algebraic independence

Corollary (Ter 2011)

The elements $\Omega, \alpha, \beta, \gamma$ are algebraically independent over $\mathbb{F}$.

The center $Z(\Delta)$

We now describe the center $Z(\Delta)$.

Theorem (Ter 2011)

Assume that q is not a root of unity. Then the algebra $Z(\Delta)$ is generated by $\Omega, \alpha, \beta, \gamma$.

Moreover $Z(\Delta)$ is isomorphic to a polynomial algebra in four variables.

Our next goal is to explain how Δ is related to the quantum group $U_q(\mathfrak{sl}_2)$.

Definition

The $\mathbb{F}$-algebra $U = U_q(\mathfrak{sl}_2)$ is defined by generators $e, f, k^{\pm 1}$ and relations

$$kk^{-1} = k^{-1}k = 1,$$
$$ke = q^2 ek, \qquad kf = q^{-2} fk,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

We call $e, f, k^{\pm 1}$ the **Chevalley generators** for U.

Irreducible modules for $U_q(\mathfrak{sl}_2)$

We review the finite-dimensional irreducible modules for $U_q(\mathfrak{sl}_2)$.

Lemma

For all integers $d \geq 0$ and $\varepsilon \in \{1, -1\}$ there exists a U-module $V_{d,\varepsilon}$ with the following property: $V_{d,\varepsilon}$ has a basis $\{v_i\}_{i=0}^d$ such that

$$\begin{aligned} kv_i &= \varepsilon q^{d-2i} v_i \qquad (0 \leq i \leq d), \\ fv_i &= [i+1]_q v_{i+1} \qquad (0 \leq i \leq d-1), \qquad fv_d = 0, \\ ev_i &= \varepsilon [d-i+1]_q v_{i-1} \qquad (1 \leq i \leq d), \qquad ev_0 = 0. \end{aligned}$$

The U-module $V_{d,\varepsilon}$ is irreducible provided that q is not a root of unity.

Earlier we gave a Casimir element for Δ. The algebra U also has a Casimir element, which we now recall.

Definition

Define $\Lambda \in U$ as follows:

$$\Phi = ef(q - q^{-1})^2 + q^{-1}k + qk^{-1}.$$

We call Λ the (normalized) **Casimir element** of U.

The Casimir element of $U_q(\mathfrak{sl}_2)$, cont.

The following result is well known.

Lemma

The Casimir element Λ is in the center $Z(U)$. Moreover on the U-module $V_{d,\varepsilon}$

$$\Lambda = \varepsilon(q^{d+1} + q^{-d-1})I.$$

When we defined U we used the Chevalley presentation. There is another presentation for U of interest, said to be **equitable**.

Lemma (Tatsuro Ito, Chih-wen Weng, Ter 2000)

The algebra U has a presentation by generators $x, y^{\pm 1}, z$ and relations

$$yy^{-1} = y^{-1}y = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We call $x, y^{\pm 1}, z$ the **equitable generators** for U.

The U-module $V_{d,\varepsilon}$ from the equitable point of view

In the equitable presentation the U-module $V_{d,\varepsilon}$ looks as follows.

$V_{d,\varepsilon}$ has three bases such that:

	x	y	z
basis 1	diagonal	lower bidiagonal	upper bidiagonal
basis 2	upper bidiagonal	diagonal	lower bidiagonal
basis 3	lower bidiagonal	upper bidiagonal	diagonal

In the equitable presentation of U the Casimir element looks as follows.

Lemma (Ter 2011)

The Casimir element Λ is equal to each of the following:

$$
\begin{aligned}
&qx + q^{-1}y + qz - qxyz, &\qquad& q^{-1}x + qy + q^{-1}z - q^{-1}zyx,\\
&qy + q^{-1}z + qx - qyzx, && q^{-1}y + qz + q^{-1}x - q^{-1}xzy,\\
&qz + q^{-1}x + qy - qzxy, && q^{-1}z + qx + q^{-1}y - q^{-1}yxz.
\end{aligned}
$$

We are now ready to describe how Δ is related to $U_q(\mathfrak{sl}_2)$.

Lemma (Ter 2011)

Let a, b, c denote nonzero scalars in $\mathbb{F}$. Then there exists an $\mathbb{F}$-algebra homomorpism $\Delta \to U_q(\mathfrak{sl}_2)$ that sends

$$\begin{aligned} A &\mapsto xa + ya^{-1} + \frac{xy - yx}{q - q^{-1}} bc^{-1}, \\ B &\mapsto yb + zb^{-1} + \frac{yz - zy}{q - q^{-1}} ca^{-1}, \\ C &\mapsto zc + xc^{-1} + \frac{zx - xz}{q - q^{-1}} ab^{-1}. \end{aligned}$$

The above homomorphism is not injective. To shrink the kernel we do the following.

From now on let a, b, c denote **mutually commuting indeterminates**.

Let $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ denote the $\mathbb{F}$-algebra of Laurent polynomials in a, b, c that have all coefficients in $\mathbb{F}$.

Consider the $\mathbb{F}$-algebra

$$U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}],$$

where $U = U_q(\mathfrak{sl}_2)$ and $\otimes = \otimes_{\mathbb{F}}$.

Theorem (Ter 2011)

There exists an injective $\mathbb{F}$*-algebra homomorphism* $\natural : \Delta \to U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ *that sends*

$$
\begin{aligned}
A &\mapsto x \otimes a + y \otimes a^{-1} + \frac{xy - yx}{q - q^{-1}} \otimes bc^{-1}, \\
B &\mapsto y \otimes b + z \otimes b^{-1} + \frac{yz - zy}{q - q^{-1}} \otimes ca^{-1}, \\
C &\mapsto z \otimes c + x \otimes c^{-1} + \frac{zx - xz}{q - q^{-1}} \otimes ab^{-1},
\end{aligned}
$$

where x, y, z *denote the equitable generators for* U.

Theorem (Ter 2011)

The homomorphism $\natural$ *sends*

$$\begin{aligned} \alpha &\mapsto \Lambda \otimes (a + a^{-1}) + 1 \otimes (b + b^{-1})(c + c^{-1}), \\ \beta &\mapsto \Lambda \otimes (b + b^{-1}) + 1 \otimes (c + c^{-1})(a + a^{-1}), \\ \gamma &\mapsto \Lambda \otimes (c + c^{-1}) + 1 \otimes (a + a^{-1})(b + b^{-1}), \end{aligned}$$

where Λ *denotes the Casimir element of* U.

Theorem (Ter 2011)

Under the homomorphism $\natural$ the image of the Casimir element Ω is

$$1 \otimes (q+q^{-1})^2 - \Lambda^2 \otimes 1 - 1 \otimes (a+a^{-1})^2 - 1 \otimes (b+b^{-1})^2$$
$$-1 \otimes (c+c^{-1})^2 - \Lambda \otimes (a+a^{-1})(b+b^{-1})(c+c^{-1})$$

where Λ denotes the Casimir element of U.

Δ and the universal DAHA of type $(C_1^\vee, C_1)$.

Our next goal is to describe how Δ is related to the double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$.

This is the most general DAHA of rank 1.

We will work with the “universal” version of DAHA.

For notational convenience define a four element set

$$\mathbb{I} = \{0, 1, 2, 3\}.$$

Definition

Let $\hat{H}_q$ denote the $\mathbb{F}$-algebra defined by generators $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ and relations

$$\begin{aligned} t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad & i \in \mathbb{I}, \\ t_i + t_i^{-1} \quad \text{is central} \qquad & i \in \mathbb{I}, \\ t_0 t_1 t_2 t_3 = q^{-1}. & \end{aligned}$$

We call $\hat{H}_q$ the **universal DAHA of type** $(C_1^\vee, C_1)$.

For notational convenience define

$$T_i = t_i + t_i^{-1} \qquad i \in \mathbb{I}.$$

The elements X, Y of $\hat{H}_q$

We will describe how Δ is related to $\hat{H}_q$.

To set the stage we first mention a few basic features of $\hat{H}_q$.

Define

$$X = t_3 t_0, \qquad\qquad Y = t_0 t_1.$$

Note that X, Y are invertible.

A basis for $\hat{H}_q$

Theorem (Ter 2012)

The following is a basis for the $\mathbb{F}$-vector space $\hat{H}_q$:

$$Y^i X^j t_0^k T_1^r T_2^s T_3^t \qquad i, j, k \in \mathbb{Z} \qquad r, s, t \in \mathbb{N}.$$

This can be proven using the Bergman Diamond Lemma.

A basis for $\hat{H}_q$

Corollary (Ter 2012)

The following are algebraically independent over $\mathbb{F}$:

$$t_0, \quad T_1, \quad T_2, \quad T_3.$$

Corollary (Ter 2012)

The following are algebraically independent over $\mathbb{F}$:

$$T_0, \quad T_1, \quad T_2, \quad T_3.$$

The center $Z(\hat{H}_q)$

We now describe the center $Z(\hat{H}_q)$.

Theorem (Ter 2012)

Assume that q is not a root of unity. Then the algebra $Z(\hat{H}_q)$ is generated by $\{T_i\}_{i \in I}$.

Moreover $Z(\hat{H}_q)$ is isomorphic to a polynomial algebra in four variables.

We mention some automorphisms of $\hat{H}_q$.

We start with an obvious one.

There exists an automorphism of $\hat{H}_q$ that sends

$$t_0 \mapsto t_1 \mapsto t_2 \mapsto t_3 \mapsto t_0.$$

We call this $\mathbb{Z}_4$-**symmetry**.

This symmetry sends

$$X \mapsto Y \mapsto q^{-1}X^{-1} \mapsto q^{-1}Y^{-1} \mapsto X.$$

The Artin braid group B_3

We will be discussing the Artin braid group B_3.

Definition

The group B_3 is defined by generators ρ, σ and relations $\rho^3 = \sigma^2$. For notational convenience define $\tau = \rho^3 = \sigma^2$.

There exists a group homomorphism $B_3 \to \mathrm{PSL}_2(\mathbb{Z})$ that sends $\rho \mapsto p$ and $\sigma \mapsto s$. Via this homomorphism we pull back the $\mathrm{PSL}_2(\mathbb{Z})$ action on Δ, to get a B_3 action on Δ as a group of automorphisms.

Next we explain how B_3 acts on $\hat{H}_q$ as a group of automorphisms.

Lemma

The group B_3 acts on $\hat{H}_q$ as a group of automorphisms such that $\tau(h) = t_0^{-1} h t_0$ for all $h \in \hat{H}_q$ and ρ, σ do the following:

h	t_0	t_1	t_2	t_3
$\rho(h)$	t_0	$t_0^{-1} t_3 t_0$	t_1	t_2
$\sigma(h)$	t_0	$t_0^{-1} t_3 t_0$	$t_1 t_2 t_1^{-1}$	t_1

An action of B_3 on $\hat{H}_q$, cont.

Lemma

The B_3 action on $\hat{H}_q$ does the following to the central elements $\{T_i\}_{i \in \mathbb{I}}$. The generator τ fixes every central element. The generators ρ, σ satisfy the table below.

h	T_0	T_1	T_2	T_3
$\rho(h)$	T_0	T_3	T_1	T_2
$\sigma(h)$	T_0	T_3	T_2	T_1

We are now ready to describe how Δ is related to $\hat{H}_q$.

Theorem (Ter 2012)

There exists an injective $\mathbb{F}$-algebra homomorphism $\psi : \Delta \to \hat{H}_q$ that sends

$$\begin{aligned} A &\mapsto t_1 t_0 + (t_1 t_0)^{-1}, \\ B &\mapsto t_3 t_0 + (t_3 t_0)^{-1}, \\ C &\mapsto t_2 t_0 + (t_2 t_0)^{-1}. \end{aligned}$$

Theorem (Ter 2012)

The homomorphism ψ sends

$$
\begin{aligned}
\alpha &\mapsto (q^{-1}t_0 + qt_0^{-1})(t_1 + t_1^{-1}) + (t_2 + t_2^{-1})(t_3 + t_3^{-1}),\\
\beta &\mapsto (q^{-1}t_0 + qt_0^{-1})(t_3 + t_3^{-1}) + (t_1 + t_1^{-1})(t_2 + t_2^{-1}),\\
\gamma &\mapsto (q^{-1}t_0 + qt_0^{-1})(t_2 + t_2^{-1}) + (t_3 + t_3^{-1})(t_1 + t_1^{-1}).
\end{aligned}
$$

Theorem (Ter 2012)

Under the homomorphism ψ the image of the Casimir element Ω is

$$(q+q^{-1})^2-(q^{-1}t_0+qt_0^{-1})^2-(t_1+t_1^{-1})^2-(t_2+t_2^{-1})^2$$
$$-(t_3+t_3^{-1})^2-(q^{-1}t_0+qt_0^{-1})(t_1+t_1^{-1})(t_2+t_2^{-1})(t_3+t_3^{-1}).$$

Theorem (Ter 2012)

For all $g \in B_3$ the following diagram commutes:

$$\begin{array}{ccc} \Delta & \xrightarrow{\psi} & \hat{H}_q \\ g \downarrow & & \downarrow g \\ \Delta & \xrightarrow[\psi]{} & \hat{H}_q \end{array}$$

Now consider the image of Δ under ψ.

As we will see, this image is related to the “spherical subalgebra”

$$\{h \in \hat{H}_q | t_0 h = h t_0\}.$$

The image of Δ under ψ

Theorem (Ter 2012)

Consider the image of Δ under ψ. The spherical subalgebra $\{h \in \hat{H}_q | t_0 h = h t_0\}$ is generated by this image together with $t_0^{\pm 1}, T_1, T_2, T_3$.

For notational convenience, from now on identify Δ with its image under the injection $\psi : \Delta \to \hat{H}_q$.

From this point of view

$$A = t_1 t_0 + (t_1 t_0)^{-1} = t_0 t_1 + (t_0 t_1)^{-1} = Y + Y^{-1},$$
$$B = t_3 t_0 + (t_3 t_0)^{-1} = t_0 t_3 + (t_0 t_3)^{-1} = X + X^{-1},$$
$$C = t_2 t_0 + (t_2 t_0)^{-1} = t_0 t_2 + (t_0 t_2)^{-1},$$

$$\alpha = (q^{-1} t_0 + q t_0^{-1}) T_1 + T_2 T_3,$$
$$\beta = (q^{-1} t_0 + q t_0^{-1}) T_3 + T_1 T_2,$$
$$\gamma = (q^{-1} t_0 + q t_0^{-1}) T_2 + T_3 T_1,$$

$$\Omega = (q + q^{-1})^2 - (q^{-1} t_0 + q t_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2 - (q^{-1} t_0 + q t_0^{-1}) T_1 T_2 T_3.$$

A presentation for the spherical subalgebra by generators and relations

We now give a presentation of the spherical subalgebra $\{h \in \hat{H}_q | t_0 h = h t_0\}$ by generators and relations.

This will be our last result of the talk.

A presentation for the spherical subalgebra

Theorem (Ter 2012)

The spherical subalgebra $\{h \in \hat{H}_q | t_0 h = h t_0\}$ *is presented by generators and relations in the following way. The generators are* A, B, C, $t_0^{\pm 1}$, $\{T_i\}_{i=1}^3$. *The relations assert that each of* $t_0^{\pm 1}$, $\{T_i\}_{i=1}^3$ *is central and* $t_0 t_0^{-1} = 1$, $t_0^{-1} t_0 = 1$,

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}},$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}},$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}},$$

$$q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma$$
$$= (q + q^{-1})^2 - (q^{-1}t_0 + qt_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2$$
$$-(q^{-1}t_0 + qt_0^{-1})T_1 T_2 T_3,$$

Theorem

where

$$
\begin{aligned}
\alpha &= (q^{-1}t_0 + qt_0^{-1})T_1 + T_2 T_3, \\
\beta &= (q^{-1}t_0 + qt_0^{-1})T_3 + T_1 T_2, \\
\gamma &= (q^{-1}t_0 + qt_0^{-1})T_2 + T_3 T_1.
\end{aligned}
$$

In this talk we introduced the universal Askey-Wilson algebra Δ.

We showed how each Leonard pair and Leonard triple of QRacah type yields a Δ-module.

We discussed how Δ is related to Q-polynomial distance-regular graphs of QRacah type.

We gave several bases for Δ, we described its center, and we showed how $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ as a group of automorphisms.

We described how Δ is related to $U_q(\mathfrak{sl}_2)$.

Finally we described how Δ is related to the universal DAHA of type $(C_1^\vee, C_1)$.

Thank you for your attention!

THE END

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