The S_3-symmetric tridiagonal algebra

Paul Terwilliger

University of Wisconsin-Madison

Overview

In this talk, we first review the **tridiagonal algebra** T.

We then introduce a generalization of T, called the S_3-**symmetric tridiagonal algebra** $\mathbb{T}$.

Given a Q-polynomial distance-regular graph Γ, we turn the tensor power $V^{\otimes 3}$ of the standard module V into a $\mathbb{T}$-module.

We describe a certain irreducible $\mathbb{T}$-submodule Λ of $V^{\otimes 3}$, said to be **fundamental**.

Preliminaries

We review some notation.

Let $\mathbb{F}$ denote a field.

Every vector space and tensor product, is understood to be over $\mathbb{F}$.

Every algebra without the Lie prefix, is understood to be associative, over $\mathbb{F}$, and have a multiplicative identity.

Preliminaries, cont.

Let $0 \neq q \in \mathbb{F}$.

For elements B, C in any algebra, define

$$[B, C] = BC - CB, \qquad [B, C]_q = qBC - q^{-1}CB.$$

The Tridiagonal algebra

We now recall the tridiagonal algebra.

Definition (Ter 2001)

For $\beta, \gamma, \gamma^*, \varrho, \varrho^* \in \mathbb{F}$ the algebra $T = T(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ is defined by generators A, A^* and relations

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0,$$
$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A] = 0.$$

We call T the **tridiagonal algebra**. The above relations are called the **tridiagonal relations**.

We mention some special cases of the tridiagonal algebra.

Lemma

Assume that $\mathbb{F}$ has characteristic 0. For

$$\beta = 2, \qquad \gamma = \gamma^* = 0, \qquad \varrho \neq 0, \qquad \varrho^* \neq 0$$

the tridiagonal relations become the **Dolan/Grady relations**

$$[A, [A, [A, A^*]]] = \varrho[A, A^*],$$
$$[A^*, [A^*, [A^*, A]]] = \varrho^*[A^*, A].$$

In this case, T becomes the enveloping algebra $U(O)$ for the **Onsager Lie algebra** O.

The algebra U_q^+

Lemma

For $\beta \neq \pm 2$,

$$\beta = q^2 + q^{-2}, \qquad \gamma = \gamma^* = 0, \qquad \varrho = \varrho^* = 0$$

*the tridiagonal relations become the q-***Serre relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0.$$

In this case, T becomes the **positive part** U_q^+ *of the q-***deformed enveloping algebra** $U_q(\widehat{\mathfrak{sl}}_2)$.

The q-Onsager algebra

Lemma

For $\beta \neq \pm 2$,

$$\beta = q^2 + q^{-2}, \qquad \gamma = \gamma^* = 0, \qquad \varrho = \varrho^* = -(q^2 - q^{-2})^2$$

*the tridiagonal relations become the q-***Dolan/Grady relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A],$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*].$$

*In this case, T becomes the q-***Onsager algebra** O_q.

The S_3-symmetric tridiagonal algebra

We now introduce the S_3-symmetric tridiagonal algebra.

Definition (Ter 2024)

For $\beta, \gamma, \gamma^*, \varrho, \varrho^* \in \mathbb{F}$ the algebra $\mathbb{T} = \mathbb{T}(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ is defined by generators

$$A_i, \quad A_i^* \qquad i \in \{1, 2, 3\}$$

and the following relations.

(i) For $i, j \in \{1, 2, 3\}$,

$$[A_i, A_j] = 0, \qquad [A_i^*, A_j^*] = 0.$$

(ii) For $i \in \{1, 2, 3\}$,

$$[A_i, A_i^*] = 0.$$

The S_3-symmetric tridiagonal algebra, cont.

Definition

Continued....

(iii) For distinct $i, j \in \{1, 2, 3\}$,

$$[A_i, A_i^2 A_j^* - \beta A_i A_j^* A_i + A_j^* A_i^2 - \gamma(A_i A_j^* + A_j^* A_i) - \varrho A_j^*] = 0,$$

$$[A_j^*, A_j^{*2} A_i - \beta A_j^* A_i A_j^* + A_i A_j^{*2} - \gamma^*(A_j^* A_i + A_i A_j^*) - \varrho^* A_i] = 0.$$

We call $\mathbb{T}$ the S_3-**symmetric tridiagonal algebra**.

How T and $\mathbb{T}$ are related

The algebras T and $\mathbb{T}$ are related as follows.

Lemma (Ter 2024)

For distinct $r, s \in \{1, 2, 3\}$ there exists an algebra homomorphism $T(\beta, \gamma, \gamma^*, \varrho, \varrho^*) \to \mathbb{T}(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ that sends

$$A \mapsto A_r, \qquad A^* \mapsto A^*_s.$$

This homomorphism is injective.

A module for the S_3-symmetric tridiagonal algebra

For the rest of this talk, the following assumptions are in effect.

Let the field $\mathbb{F} = \mathbb{C}$.

Let Γ denote a distance-regular graph, with vertex set X, path-length distance function ∂, and diameter $D \geq 1$.

For $x \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

Assumptions continued....

Assume that Γ is Q-polynomial, with eigenvalue sequence $\{\theta_i\}_{i=0}^D$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^D$.

Some scalars

Lemma

There exist real scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ *such that:*

(i) $\beta + 1$ *is equal to each of*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

for $2 \leq i \leq D - 1$.

(ii) *For* $1 \leq i \leq D - 1$, *both*

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \qquad \gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}.$$

(iii) *For* $1 \leq i \leq D$, *both*

$$\varrho = \theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i),$$
$$\varrho^* = \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i).$$

The standard module

Next, we recall the standard module associated with Γ.

Definition

Let V denote a vector space over $\mathbb{C}$ with basis X. We call V the **standard module** associated with Γ.

Definition

We define the vector space $V^{\otimes 3} = V \otimes V \otimes V$ and the set

$$X^{\otimes 3} = \{x \otimes y \otimes z | x, y, z \in X\}.$$

Note that $X^{\otimes 3}$ is a basis for $V^{\otimes 3}$.

We now state our first main result.

A module for the S_3-symmetric tridiagonal algebra

Theorem (Ter 2024)

For the above scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ the vector space $V^{\otimes 3}$ becomes a $\mathbb{T}(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$-module on which the generators $\{A_i\}_{i=1}^3$, $\{A_i^*\}_{i=1}^3$ act as follows. For $x, y, z \in X$,

$$A_1(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(x)} \xi \otimes y \otimes z,$$

$$A_2(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(y)} x \otimes \xi \otimes z,$$

$$A_3(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(z)} x \otimes y \otimes \xi,$$

$$A_1^*(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(y,z)},$$

$$A_2^*(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(z,x)},$$

$$A_3^*(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(x,y)}.$$

Node actions and edge actions

The above six actions are discussed in Bill Martin's work on **Scaffolds**.

The actions of A_1, A_2, A_3 are called **node actions**, and the actions of A_1^*, A_2^*, A_3^* are called **edge actions**.

The fundamental $\mathbb{T}$-submodule Λ

Next, we discuss a certain irreducible $\mathbb{T}$-submodule $\Lambda \subseteq V^{\otimes 3}$, said to be **fundamental**.

We bring in some notation.

Let $A : V \to V$ denote the adjacency map for Γ:

$$Ax = \sum_{\xi \in \Gamma(x)} \xi \qquad (x \in X).$$

For $0 \leq i \leq D$ let E_i denote the primitive idempotent of A for θ_i.

Let $(\, , \,)$ denote the unique Hermitean form $V \times V \to \mathbb{C}$ with respect to which the basis X is orthonormal.

Lemma

The following hold.

(i) *There exists a unique Hermitean form* $\langle \, , \, \rangle : V^{\otimes 3} \times V^{\otimes 3} \to \mathbb{C}$ *with respect to which the basis* $X^{\otimes 3}$ *is orthonormal.*

(ii) *For* $u, v, w, u', v', w' \in V$ *we have*

$$\langle u \otimes v \otimes w, u' \otimes v' \otimes w' \rangle = (u, u')(v, v')(w, w').$$

The $\mathbb{T}$-action

The Hermitean form $\langle\,,\,\rangle$ respects the $\mathbb{T}$-action as follows.

Lemma

For $r \in \{1,2,3\}$ and $u, v \in V^{\otimes 3}$ we have

$$\langle A^{(r)}u, v\rangle = \langle u, A^{(r)}v\rangle, \qquad \langle A^{*(r)}u, v\rangle = \langle u, A^{*(r)}v\rangle.$$

Irreducible $\mathbb{T}$-modules

Definition

A $\mathbb{T}$-module W is called **irreducible** whenever $W \neq 0$ and W does not contain a $\mathbb{T}$-submodule besides 0 and W.

Lemma

The $\mathbb{T}$-module $V^{\otimes 3}$ is an orthogonal direct sum of irreducible $\mathbb{T}$-submodules.

The vector $\mathbf{1}$

The standard module V contains the vector

$$\mathbf{1} = \sum_{x \in X} x.$$

We abbreviate $\mathbf{1}^{\otimes 3} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ and note that

$$\mathbf{1}^{\otimes 3} = \sum_{x,y,z \in X} x \otimes y \otimes z.$$

The fundamental $\mathbb{T}$-submodule Λ

Lemma (Ter 2024)

There exists a unique irreducible $\mathbb{T}$-submodule Λ of $V^{\otimes 3}$ that contains $\mathbf{1}^{\otimes 3}$.

Definition (Ter 2024)

The $\mathbb{T}$-submodule Λ is called **fundamental**.

In order to describe Λ, we display some vectors that it contains.

Definition

For $0 \leq h, i, j \leq D$ define

$$P_{h,i,j} = \sum_{\substack{x,y,z \in X \\ \partial(y,z)=h \\ \partial(z,x)=i \\ \partial(x,y)=j}} x \otimes y \otimes z.$$

For example,

$$P_{0,0,0} = \sum_{x \in X} x \otimes x \otimes x.$$

Lemma

We have

$$P_{h,i,j} \in \Lambda, \qquad (0 \leq h, i, j \leq D).$$

The vectors $P_{h,i,j}$ cont.

Recall the intersection numbers $p^h_{i,j}$ and valencies k_h for Γ.

Lemma

The following vectors are mutually orthogonal:

$$P_{h,i,j} \qquad\qquad 0 \leq h, i, j \leq D.$$

For $0 \leq h, i, j \leq D$ *we have*

$$\|P_{h,i,j}\|^2 = |X| k_h p^h_{i,j}.$$

Lemma

$P_{h,i,j} = 0$ *if and only if* $p^h_{i,j} = 0$ $(0 \leq h, i, j \leq D)$.

The vectors $Q_{h,i,j}$

Definition (Cameron, Goethals, Seidel 1978)

For $0 \leq h, i, j \leq D$ define

$$Q_{h,i,j} = |X| \sum_{x \in X} E_h x \otimes E_i x \otimes E_j x.$$

The vectors $Q_{h,i,j}$ cont.

Lemma

We have

$$Q_{h,i,j} \in \Lambda, \qquad (0 \le h, i, j \le D).$$

The vectors $Q_{h,i,j}$ cont.

Recall the Krein parameters $q^h_{i,j}$ and multiplicities m_h for Γ.

Lemma (Cameron, Goethals, Seidel 1978)

The following vectors are mutually orthogonal:

$$Q_{h,i,j} \qquad\qquad 0 \leq h, i, j \leq D.$$

For $0 \leq h, i, j \leq D$ *we have*

$$\|Q_{h,i,j}\|^2 = |X| m_h q^h_{i,j}.$$

Lemma (Cameron, Goethals, Seidel 1978)

$Q_{h,i,j} = 0$ *if and only if* $q^h_{i,j} = 0$ $(0 \leq h, i, j \leq D)$.

Two commuting actions

For the rest of this talk, let G denote a subgroup of $\mathrm{Aut}(\Gamma)$.

We are going to display a G action on $V^{\otimes 3}$ that commutes with the $\mathbb{T}$ action.

The G action on V

We recall how V becomes a G-module.

Pick $v \in V$ and write $v = \sum_{x \in X} v_x x$ $(v_x \in \mathbb{C})$. For all $g \in G$,

$$g(v) = \sum_{x \in X} v_x g(x).$$

Since g respects adjacency, we have $gA = Ag$ on V.

The G action on $V^{\otimes 3}$

Next, we describe how $V^{\otimes 3}$ becomes a G-module.

For $u, v, w \in V$ and $g \in G$ we have

$$g(u \otimes v \otimes w) = g(u) \otimes g(v) \otimes g(w).$$

The actions of G and $\mathbb{T}$ commute

Lemma

For $g \in G$ and $B \in \mathbb{T}$, we have $gB = Bg$ on $V^{\otimes 3}$.

The $\mathbb{T}$-submodule $\mathrm{Fix}(G)$

Definition

We define the set

$$\mathrm{Fix}(G) = \{v \in V^{\otimes 3} | g(v) = v \, \forall g \in G\}.$$

Lemma (Ter 2024)

$\mathrm{Fix}(G)$ *is a* $\mathbb{T}$*-submodule of* $V^{\otimes 3}$ *that contains* Λ.

A basis for $\mathrm{Fix}(G)$

Next, we display a basis for $\mathrm{Fix}(G)$.

Recall that $V^{\otimes 3}$ has an orthonormal basis $X^{\otimes 3}$.

The group G acts on the set $X^{\otimes 3}$.

A basis for $\mathrm{Fix}(G)$, cont.

Definition

Referring to the G action on the set $X^{\otimes 3}$, let $\mathcal{O}$ denote the set of orbits. For each orbit $\Omega \in \mathcal{O}$ define

$$\chi_\Omega = \sum_{x \otimes y \otimes z \in \Omega} x \otimes y \otimes z.$$

We call χ_Ω the **characteristic vector** of Ω.

A basis for $\mathrm{Fix}(G)$, cont.

Lemma

The following is an orthogonal basis for the vector space $\mathrm{Fix}(G)$*:*

$$\chi_\Omega, \qquad \Omega \in \mathcal{O}.$$

We mentioned that $\Lambda \subseteq \mathrm{Fix}(G)$.

Next, we give an example for which $\Lambda = \mathrm{Fix}(G)$.

The Hamming graph $H(D, N)$

Theorem

Assume that Γ is the Hamming graph $H(D, N)$ with $D \geq 1$ and $N \geq 3$. Then for $G = \mathrm{Aut}(\Gamma)$,

$$\Lambda = \mathrm{Fix}(G).$$

Moreover

$$\dim \Lambda = \binom{D+4}{4}.$$

Summary

In this talk, we introduced the S_3-**symmetric tridiagonal algebra** $\mathbb{T}$.

For a Q-**polynomial distance-regular graph** Γ, we turned the tensor power $V^{\otimes 3}$ of the standard module V into a $\mathbb{T}$-module.

We identified a certain irreducible $\mathbb{T}$-submodule Λ of $V^{\otimes 3}$, said to be **fundamental**.

We described some vectors $P_{h,i,j}$ and $Q_{h,i,j}$ in Λ.

For a subgroup G of $\mathrm{Aut}(\Gamma)$ we described a $\mathbb{T}$-submodule $\mathrm{Fix}(G)$ of $V^{\otimes 3}$ that contains Λ.

For $\Gamma = H(D, N)$ and $G = \mathrm{Aut}(\Gamma)$ we showed that $\Lambda = \mathrm{Fix}(G)$.

THANK YOU FOR YOUR ATTENTION!