The alternating PBW basis for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Overview

The positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$ has a presentation with two generators A, B that satisfy the cubic q-Serre relations.

We introduce a PBW basis for U_q^+, said to be alternating.

Each element of this PBW basis commutes with exactly one of A, B, $qAB - q^{-1}BA$.

This gives three types of PBW basis elements; the elements of each type mutually commute.

We interpret the alternating PBW basis in terms of a q-shuffle algebra associated with affine $\mathfrak{sl}_2$.

We show how the alternating PBW basis is related to the PBW basis for U_q^+ found by Damiani in 1993.

Acknowledgement

Our discovery of the alternating PBW basis was inspired by the work of mathematical physicists P. Baseilhac, K. Koizumi, K. Shigechi concerning boundary integrable systems with hidden symmetries.

We were led to the alternating PBW basis while trying to understand their work.

Simply put, our discovery would not have occurred without their inspiration.

Preliminaries

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Fix a field $\mathbb{F}$.

Each vector space discussed is over $\mathbb{F}$.

Each algebra discussed is associative, over $\mathbb{F}$, and has a 1.

PBW bases

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a **Poincaré-Birkhoff-Witt** (or **PBW**) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

For elements X, Y in any algebra, define their **commutator** and q-**commutator** by

$$[X, Y] = XY - YX, \qquad\qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_qX^2YX + [3]_qXYX^2 - YX^3.$$

The algebra U_q^+

Definition

Define the algebra U_q^+ by generators A, B and relations

$$[A,[A,[A,B]_q]_{q^{-1}}] = 0,$$
$$[B,[B,[B,A]_q]_{q^{-1}}] = 0.$$

We call U_q^+ the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

The above relations are called the **q-Serre relations**.

Why we care about U_q^+

We briefly explain why U_q^+ is of interest.

Let V denote a finite-dimensional irreducible U_q^+-module on which A, B are diagonalizable. Then:

- the eigenvalues of A and B on V have the form

$$
\begin{array}{lll}
A: & \{aq^{d-2i}\}_{i=0}^{d} & 0 \neq a \in \mathbb{F}, \\
B: & \{bq^{d-2i}\}_{i=0}^{d} & 0 \neq b \in \mathbb{F}.
\end{array}
$$

- For $0 \leq i \leq d$ let V_i (resp. V_i^*) denote the eigenspace of A (resp. B) for the eigenvalue aq^{d-2i} (resp. bq^{d-2i}). Then

$$
BV_i \subseteq V_{i-1} + V_i + V_{i+1},
$$
$$
AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*,
$$

where $V_{-1} = 0 = V_{d+1}$ and $V_{-1}^* = 0 = V_{d+1}^*$.

Consequently A, B act on V as a **tridiagonal pair**.

The topic of tridiagonal pairs is an active area of research, with links to

- combinatorics and graph theory (E. Bannai, T. Ito, W. Martin, S. Miklavic, K. Nomura, A. Pascasio, H. Tanaka) ;
- special functions and orthogonal polynomials (H. Alnajjar, B. Curtin, A. Grunbaum, E. Hanson, M. Ismail, J. H. Lee, R. Vidunas);
- quantum groups and representation theory (S. Bockting-Conrad, H. W. Huang, S. Kolb);
- mathematical physics (P. Baseilhac, S. Belliard, L. Vinet, A. Zhedanov)

We now return to U_q^+.

An $\mathbb{N}^2$-grading for U_q^+

The algebra U_q^+ has a grading that we now describe.

Note that the q-Serre relations are homogeneous in both A and B.

Therefore the algebra U_q^+ has an $\mathbb{N}^2$-grading such that A and B are homogeneous, with degrees $(1,0)$ and $(0,1)$ respectively.

For $(i,j) \in \mathbb{N}^2$ let $d_{i,j}$ denote the dimension of the (i,j)-homogeneous component of U_q^+.

These dimensions are described by the generating function

$$\sum_{(i,j)\in\mathbb{N}^2} d_{i,j}\lambda^i\mu^j = \prod_{\ell=1}^{\infty} \frac{1}{1-\lambda^\ell\mu^{\ell-1}} \frac{1}{1-\lambda^\ell\mu^\ell} \frac{1}{1-\lambda^{\ell-1}\mu^\ell}.$$

An $\mathbb{N}^2$-grading of U_q^+, cont.

For $0 \leq i, j \leq 6$ the dimension $d_{i,j}$ is given in the (i,j)-entry of the matrix below:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 6 & 8 & 9 & 9 & 9 \\ 1 & 3 & 8 & 14 & 19 & 21 & 22 \\ 1 & 3 & 9 & 19 & 32 & 42 & 48 \\ 1 & 3 & 9 & 21 & 42 & 66 & 87 \\ 1 & 3 & 9 & 22 & 48 & 87 & 134 \end{pmatrix}$$

A PBW basis for U_q^+

In 1993, Damiani obtained a PBW basis for U_q^+, involving some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{E_{n\delta}\}_{n=1}^{\infty}. \qquad (1)$$

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \qquad E_{\alpha_1} = B, \qquad E_{\delta} = q^{-2}BA - AB,$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}},$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}A - AE_{(n-1)\delta+\alpha_1}.$$

A PBW basis for U_q^+, cont.

The elements (1) are homogeneous with degrees shown below:

element	degree
$E_{n\delta+\alpha_0}$	$(n+1, n)$
$E_{n\delta+\alpha_1}$	$(n, n+1)$
$E_{n\delta}$	(n, n)

Theorem (Damiani 1993)

A PBW basis for U_q^+ is given by the elements (1) in linear order

$$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots$$
$$\cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots$$
$$\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.$$

Moreover the elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

The Damiani PBW basis in closed form

The Damiani PBW basis elements are defined recursively.

Next we describe these elements in closed form, using a q-shuffle algebra.

For this q-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described on the next slide.

The free algebra $\mathbb{V}$

Let x, y denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators x, y.

By a **letter** in $\mathbb{V}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words; this basis is called **standard**.

The q-shuffle product on $\mathbb{V}$

We just defined the free algebra $\mathbb{V}$.

There is another algebra structure on $\mathbb{V}$, called the q-**shuffle algebra**. This is due to M. Rosso 1995.

The q-shuffle product will be denoted by $\star$.

The q-shuffle product on $\mathbb{V}$, cont.

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v \rangle}$$

where

$\langle\, , \,\rangle$	x	y
x	2	-2
y	-2	2

So

$$x \star y = xy + q^{-2} yx, \qquad y \star x = yx + q^{-2} xy,$$
$$x \star x = (1+q^2)xx, \qquad y \star y = (1+q^2)yy.$$

The q-shuffle product on $\mathbb{V}$, cont.

For words u, v in $\mathbb{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$
\begin{aligned}
u \star v = {} & a_1 a_2 b_1 b_2 \\
& + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\
& + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\
& + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\
& + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\
& + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle}
\end{aligned}
$$

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{V}$ into an algebra.

Definition

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by x, y.

The algebra U is described as follows. We have

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

So in the q-shuffle algebra $\mathbb{V}$ the elements x, y satisfy the q-Serre relations.

Consequently there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$.

The map $\natural$ has image U by construction.

How U_q^+ is related to U.

Theorem (Rosso, 1995)

The map $\natural : U_q^+ \to U$ is an algebra isomorphism.

Next we describe how the map $\natural$ acts on the Damiani PBW basis for U_q^+.

The Catalan words in $\mathbb{V}$

Give each letter x, y a weight:

$$\overline{x} = 1, \qquad \overline{y} = -1.$$

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is **Catalan** whenever $\overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_i$ is nonnegative for $1 \leq i \leq n-1$ and zero for $i = n$. In this case n is even.

Example

For $0 \leq n \leq 3$ we give the Catalan words of length $2n$.

n	Catalan words of length $2n$
0	1
1	xy
2	$xyxy, \quad xxyy$
3	$xyxyxy, \quad xxyyxy, \quad xyxxyy, \quad xxyxyy, \quad xxxyyy$

Definition

For $n \in \mathbb{N}$ define

$$C_n = \sum v_1 v_2 \cdots v_{2n} [1]_q [1 + \overline{v}_1]_q [1 + \overline{v}_1 + \overline{v}_2]_q \cdots [1 + \overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_{2n}]_q,$$

where the sum is over all the Catalan words $v_1 v_2 \cdots v_{2n}$ in $\mathbb{V}$ that have length $2n$.

Example

We have

$$C_0 = 1, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [3]_q [2]_q^2 xxyy,$$

$$C_3 = [2]_q^3 xyxyxy + [3]_q [2]_q^3 xxyyxy + [3]_q [2]_q^3 xyxxyy + [3]_q^2 [2]_q^3 xxyxyy + [4]_q [3]_q^2 [2]_q^2 xxxyyy.$$

The Damiani PBW basis in closed form, cont.

Theorem (Terwilliger 2018)

The map $\natural$ *sends*

$$E_{n\delta+\alpha_0} \mapsto q^{-2n}(q-q^{-1})^{2n} x C_n,$$
$$E_{n\delta+\alpha_1} \mapsto q^{-2n}(q-q^{-1})^{2n} C_n y$$

for $n \geq 0$*, and*

$$E_{n\delta} \mapsto -q^{-2n}(q-q^{-1})^{2n-1} C_n$$

for $n \geq 1$*.*

The $\{C_n\}_{n=1}^{\infty}$ mutually commute

We mentioned earlier that $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

Corollary

For $i, j \in \mathbb{N}$,

$$C_i \star C_j = C_j \star C_i.$$

The alternating PBW basis for U_q^+

We just described the Damiani PBW basis for U_q^+.

In the coming slides, we will obtain another PBW basis for U_q^+, said to be alternating.

This PBW basis has the following features.

The alternating PBW basis for U_q^+, cont.

For the alternating PBW basis,

- each element commutes with exactly one of

$$A, \qquad B, \qquad [A, B]_q;$$

- the elements that commute with A mutually commute;
- the elements that commute with B mutually commute;
- the elements that commute with $[A, B]_q$ mutually commute.

From now on, we identify U_q^+ with U, via the isomorphism $\natural$.

Definition

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is called **alternating** whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus an alternating word has the form $\cdots xyxy \cdots$.

The alternating words, cont.

Definition

We name the alternating words as follows:

$$\begin{array}{llll} W_0 = x, & W_{-1} = xyx, & W_{-2} = xyxyx, & \ldots \\ W_1 = y, & W_2 = yxy, & W_3 = yxyxy, & \ldots \\ G_1 = yx, & G_2 = yxyx, & G_3 = yxyxyx, & \ldots \\ \tilde{G}_1 = xy, & \tilde{G}_2 = xyxy, & \tilde{G}_3 = xyxyxy, & \ldots \end{array}$$

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

The alternating words, cont.

For $k \in \mathbb{N}$,

name	description	x-degree	y-degree	length
W_{-k}	$xyxy \cdots x$	$k+1$	k	$2k+1$
W_{k+1}	$yxyx \cdots y$	k	$k+1$	$2k+1$
G_k	$yxyx \cdots x$	k	k	$2k$
$\tilde{G}_k$	$xyxy \cdots y$	k	k	$2k$

U contains the alternating words

We are going to show that U contains the alternating words.

As a warmup, consider the alternating words xy and yx.

One checks

$$xy = q\frac{qx \star y - q^{-1}y \star x}{q^2 - q^{-2}}, \qquad yx = q\frac{qy \star x - q^{-1}x \star y}{q^2 - q^{-2}}.$$

Therefore U contains xy and yx.

Comment on the alternating words

Over the next four slides, we display many relations satisfied by the alternating words.

These relations will imply that U contains the alternating words.

Relations for the alternating words, I

Lemma

For $k \in \mathbb{N}$ the following holds in the q-shuffle algebra $\mathbb{V}$:

$$
\begin{aligned}
[W_0, W_{k+1}] = [W_{-k}, W_1] &= (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \\
[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q &= (q - q^{-1}) W_{-k-1}, \\
[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q &= (q - q^{-1}) W_{k+2}.
\end{aligned}
$$

Lemma

For $k, \ell \in \mathbb{N}$ the following relations hold in the q-shuffle algebra $\mathbb{V}$:

$$
\begin{gathered}
[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \\
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \\
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \\
[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \\
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \\
[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \\
[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \\
[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.
\end{gathered}
$$

Relations for the alternating words, III

Lemma

For $k, \ell \in \mathbb{N}$ the following relations hold in the q-shuffle algebra $\mathbb{V}$:

$$
\begin{aligned}
&[W_{-k}, G_\ell]_q = [W_{-\ell}, G_k]_q, \qquad [G_k, W_{\ell+1}]_q = [G_\ell, W_{k+1}]_q,\\
&[\tilde{G}_k, W_{-\ell}]_q = [\tilde{G}_\ell, W_{-k}]_q, \qquad [W_{\ell+1}, \tilde{G}_k]_q = [W_{k+1}, \tilde{G}_\ell]_q,\\
&[G_k, \tilde{G}_{\ell+1}] - [G_\ell, \tilde{G}_{k+1}] = q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q,\\
&[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_\ell, G_{k+1}] = q[W_{\ell+1}, W_{-k}]_q - q[W_{k+1}, W_{-\ell}]_q,\\
&[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q = q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}],\\
&[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q = q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}].
\end{aligned}
$$

Relations for the alternating words, IV

Lemma

For $n \geq 1$,

$$\sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{2k+1-n},$$

$$\sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{2k+1-n}.$$

Obtaining the alternating words from x, y

Lemma

Using the equations below, the alternating words in $\mathbb{V}$ are recursively obtained from x, y in the following order:

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad \ldots$$

We have $W_0 = x$ and $W_1 = y$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \star \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n \star W_0 - W_0 \star W_n}{(1 + q^{-2n})(1 - q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 \star W_n - W_n \star W_0}{1 - q^{-2}},$$

$$W_{-n} = \frac{q W_0 \star G_n - q^{-1} G_n \star W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{q G_n \star W_1 - q^{-1} W_1 \star G_n}{q - q^{-1}}.$$

U contains the alternating words

Corollary

U contains the alternating words.

We will use the alternating words to obtain a PBW basis for U.

We wont use all the alternating words, because some of them can be written in terms of the others.

Over the next five slides, we show how to write $\{G_n\}_{n=1}^{\infty}$ in terms of $\{W_{-n}\}_{n=0}^{\infty}$, $\{W_{n+1}\}_{n=0}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$.

A change of variables

At this point, it is convenient to make a change of variables.

Definition

Define elements $\{D_n\}_{n=0}^{\infty}$ in $\mathbb{V}$ such that $D_0 = 1$ and for $n \geq 1$,

$$D_0 \star \tilde{G}_n + D_1 \star \tilde{G}_{n-1} + \cdots + D_n \star \tilde{G}_0 = 0.$$

A change of variables, cont.

Example

We have

$$D_1 = -\tilde{G}_1,$$
$$D_2 = \tilde{G}_1 \star \tilde{G}_1 - \tilde{G}_2,$$
$$D_3 = 2\tilde{G}_1 \star \tilde{G}_2 - \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 - \tilde{G}_3,$$
$$D_4 = \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 + 2\tilde{G}_1 \star \tilde{G}_3 + \tilde{G}_2 \star \tilde{G}_2 - 3\tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_2 - \tilde{G}_4$$

and

$$\tilde{G}_1 = -D_1,$$
$$\tilde{G}_2 = D_1 \star D_1 - D_2,$$
$$\tilde{G}_3 = 2D_1 \star D_2 - D_1 \star D_1 \star D_1 - D_3,$$
$$\tilde{G}_4 = D_1 \star D_1 \star D_1 \star D_1 + 2D_1 \star D_3 + D_2 \star D_2 - 3D_1 \star D_1 \star D_2 - D_4.$$

A change of variables, cont.

The following two results clarify how the D_n are related to the $\tilde{G}_n$.

Lemma

For $n \geq 1$ the following hold in the q-shuffle algebra $\mathbb{V}$.

(i) *D_n is a homogeneous polynomial in $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$ that has total degree n, where we view each $\tilde{G}_i$ as having degree i.*

(ii) *$\tilde{G}_n$ is a homogeneous polynomial in $D_1, D_2, \ldots, D_n$ that has total degree n, where we view each D_i as having degree i.*

A change of variables, cont.

Lemma

The following coincide:

(i) *the subalgebra of the q-shuffle algebra* $\mathbb{V}$ *generated by* $\{D_n\}_{n=1}^{\infty}$;

(ii) *the subalgebra of the q-shuffle algebra* $\mathbb{V}$ *generated by* $\{\tilde{G}_n\}_{n=1}^{\infty}$.

Eliminating the G_n

Using our earlier relations I–IV we obtain the following result.

Lemma

For $n \in \mathbb{N}$ we have

$$G_n = q^{2n} D_n + q^2 \sum_{\substack{i+j+k+1=n \\ i,j,k \geq 0}} W_{-i} \star D_j \star W_{k+1}.$$

Because of this result, we eliminate the G_n from consideration, as we construct the alternating PBW basis.

The alternating PBW basis for U

Theorem

A PBW basis for U is obtained by the elements

$$\{W_{-i}\}_{i\in\mathbb{N}}, \qquad \{\tilde{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \qquad i,j,k \in \mathbb{N}.$$

Definition

The above PBW basis is called **alternating**.

Comparing the Damiani PBW basis and the alternating PBW basis

Next we explain how the Damiani PBW basis is related to the alternating PBW basis.

We adopt the following point of view. Instead of working directly with the Damiani PBW basis elements, we will work with the closely related elements $\{xC_n\}_{n=0}^{\infty}$, $\{C_n y\}_{n=0}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$.

Comparing $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$

Earlier we saw how the elements $\{\tilde{G}_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ are related.

We now explain how these elements are related to $\{C_n\}_{n=1}^{\infty}$.

Theorem

For $n \geq 1$,

$$C_n = (-1)^n \sum_{i=0}^{n} q^{2i-n} D_i \star D_{n-i}$$

Comparing $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$, cont.

Corollary

For $n \geq 1$ the following hold in the q-shuffle algebra $\mathbb{V}$.

(i) *C_n is a homogeneous polynomial in $D_1, D_2, \ldots, D_n$ that has total degree n, where we view each D_i as having degree i.*

(ii) *D_n is a homogeneous polynomial in $C_1, C_2, \ldots, C_n$ that has total degree n, where we view each C_i as having degree i.*

Comparing $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$, cont.

Corollary

For $n \geq 1$ the following hold in the q-shuffle algebra $\mathbb{V}$.

(i) *C_n is a homogeneous polynomial in $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$ that has total degree n, where we view each $\tilde{G}_i$ as having degree i.*

(ii) *$\tilde{G}_n$ is a homogeneous polynomial in $C_1, C_2, \ldots, C_n$ that has total degree n, where we view each C_i as having degree i.*

Corollary

The following (i)–(iii) *coincide:*

(i) *the subalgebra of the q-shuffle algebra* $\mathbb{V}$ *generated by* $\{C_n\}_{n=1}^{\infty}$*;*

(ii) *the subalgebra of the q-shuffle algebra* $\mathbb{V}$ *generated by* $\{D_n\}_{n=1}^{\infty}$*;*

(iii) *the subalgebra of the q-shuffle algebra* $\mathbb{V}$ *generated by* $\{\tilde{G}_n\}_{n=1}^{\infty}$.

Comparing the Damiani PBW basis and the alternating PBW basis

We now write the $\{xC_n\}_{n=0}^{\infty}$ and $\{C_n y\}_{n=0}^{\infty}$ in the alternating PBW basis.

Theorem

For $n \in \mathbb{N}$,

$$xC_n = (-1)^n q^{-n} \sum_{i=0}^{n} W_{-i} \star D_{n-i},$$

$$C_n y = (-1)^n q^{-n} \sum_{i=0}^{n} D_{n-i} \star W_{i+1}.$$

Comparing the Damiani PBW basis and the alternating PBW basis, cont.

We now write the $\{W_{-n}\}_{n=0}^{\infty}$ and $\{W_{n+1}\}_{n=0}^{\infty}$ in the Damiani PBW basis.

Theorem

For $n \in \mathbb{N}$,

$$W_{-n} = \sum_{i=0}^{n} (-1)^i q^i (xC_i) \star \tilde{G}_{n-i},$$

$$W_{n+1} = \sum_{i=0}^{n} (-1)^i q^i \tilde{G}_{n-i} \star (C_i y).$$

Summary

First we recalled the Damiani PBW basis for U_q^+.

Next we expressed the Damiani PBW basis elements in closed form, using a q-shuffle algebra.

Using the q-shuffle algebra, we obtained an attractive new PBW basis for U_q^+, said to be alternating.

Finally we described how the Damiani PBW basis is related to the alternating PBW basis.

THANK YOU FOR YOUR ATTENTION!