The $\mathbb{Z}_3$-Symmetric Down-Up algebra

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Memorial

In memory of Georgia Benkart (1947–2022).

She was my valued colleague for many years, and an international leader in Lie theory and quantum groups.

Overview

In 1998, Georgia Benkart and Tom Roby introduced the **down-up algebra** $\mathcal{A}$.

This algebra is defined by two generators and two relations called the **down-up relations**.

In this talk, we introduce the $\mathbb{Z}_3$-**symmetric down-up algebra** $\mathbb{A}$.

After some motivations, we will discuss how $\mathbb{A}$ is related to some familiar algebras in the literature.

Definition (Benkart and Roby 1998)

For $\alpha, \beta, \gamma \in \mathbb{C}$ the algebra $\mathcal{A} = \mathcal{A}(\alpha, \beta, \gamma)$ is defined by generators A, B and the following relations:

$$BA^2 = \alpha ABA + \beta A^2 B + \gamma A,$$
$$B^2 A = \alpha BAB + \beta AB^2 + \gamma B.$$

The algebra $\mathcal{A}$ is called the **down-up algebra** with parameters α, β, γ.

Motivation: The Down-Up algebra and posets

The down-up algebra arises in the study of **partially ordered sets**.

I will illustrate with one example, called the **Attenuated Space poset** $\mathcal{A}_q(N, M)$.

Motivation: The Attenuated Space poset

Let $\mathbb{F}_q$ denote a finite field with q elements.

Let N, M denote positive integers.

Let H denote a vector space over $\mathbb{F}_q$ that has dimension $N + M$.

Fix a subspace $h \subseteq H$ that has dimension M.

Motivation: The Attenuated Space poset, cont.

Let the set X consist of the subspaces of H that have zero intersection with h.

The set X, together with the inclusion relation, is a poset denoted by $\mathcal{A}_q(N, M)$ and called the **Attenuated Space poset**.

The poset $\mathcal{A}_q(N, M)$ is ranked with rank N; the rank of a vertex is equal to its dimension.

For $\mathcal{A}_q(N, M)$ we now define some raising and lowering maps.

Let V denote a vector space over $\mathbb{C}$ that has basis X.

We call V the **standard module** for $\mathcal{A}_q(N, M)$.

The raising map for $\mathcal{A}_q(N, M)$

Definition

Define a linear map $R : V \to V$ such that for $y \in X$,

$$Ry = \sum_{z \text{ covers } y} z.$$

We call R the **raising map** for $\mathcal{A}_q(N, M)$.

Definition (S. Ghosh and M. Srinivasan 2022)

Define a linear map $L : V \to V$ such that for $z \in X$,

$$Lz = \sum_{z \text{ covers } y} q^{\dim y} y.$$

We call L the q-**lowering map** for $\mathcal{A}_q(N, M)$.

The maps R, L satisfy the down-up relations

Theorem (Ter 1990)

For $\mathcal{A}_q(N, M)$ the maps R, L satisfy the down-up relations:

$$L^2R - q(q+1)LRL + q^3RL^2 = -q^{N+M}(q+1)L,$$
$$LR^2 - q(q+1)RLR + q^3R^2L = -q^{N+M}(q+1)R.$$

We have been discussing the poset $\mathcal{A}_q(N, M)$.

There is a family of posets with similar features, said to be **uniform**.

See the paper

P. Terwilliger. The incidence algebra of a uniform poset (1990).

The $\mathbb{Z}_3$-symmetric down-up algebra

We now define the $\mathbb{Z}_3$-symmetric down-up algebra.

Definition (Ter 2023)

For $\alpha, \beta, \gamma \in \mathbb{C}$ the algebra $\mathbb{A} = \mathbb{A}(\alpha, \beta, \gamma)$ is defined by generators A, B, C and the following relations:

$$BA^2 = \alpha ABA + \beta A^2 B + \gamma A, \quad B^2 A = \alpha BAB + \beta AB^2 + \gamma B,$$
$$CB^2 = \alpha BCB + \beta B^2 C + \gamma B, \quad C^2 B = \alpha CBC + \beta BC^2 + \gamma C,$$
$$AC^2 = \alpha CAC + \beta C^2 A + \gamma C, \quad A^2 C = \alpha ACA + \beta CA^2 + \gamma A.$$

We call $\mathbb{A}$ the $\mathbb{Z}_3$-**symmetric down-up algebra** with parameters α, β, γ.

The algebras $\mathcal{A}$ and $\mathbb{A}$ are related as follows.

Lemma

For $\alpha, \beta, \gamma \in \mathbb{C}$ there exists an algebra homomorphism $\mathcal{A}(\alpha, \beta, \gamma) \to \mathbb{A}(\alpha, \beta, \gamma)$ that sends $A \mapsto A$ and $B \mapsto B$.

The above homomorphism might or might not be injective, depending on the values α, β, γ.

An open problem

The following problem is open.

Problem

Determine the values of α, β, γ such that the map $\mathcal{A}(\alpha, \beta, \gamma) \to \mathbb{A}(\alpha, \beta, \gamma)$ is injective.

Motivation for $\mathbb{A}$: LR triples

The algebra $\mathbb{A}(\alpha, \beta, \gamma)$ was motivated by the concept of a **lowering-raising triple** (or *LR* **triple**).

An LR triple is defined as follows.

Let $\mathbb{F}$ denote any field, and fix an integer $d \geq 1$.

Let V denote a vector space over $\mathbb{F}$ with dimension $d + 1$.

By a **decomposition of** V we mean a sequence $\{V_i\}_{i=0}^{d}$ of one dimensional subspaces whose direct sum is V.

We represent this decomposition by a sequence of dots:

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ V_0 & V_1 & & \cdots & \cdots & V_d \end{array}$$

Lowering and Raising maps

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V.

Consider a linear transformation $A \in \mathrm{End}(V)$.

We say that A **lowers** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$.

We say that A **raises** $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $AV_d = 0$.

Lowering and Raising maps, cont.

$$\leftarrow A \text{ lowers}$$

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$$

$$A \text{ raises} \rightarrow$$

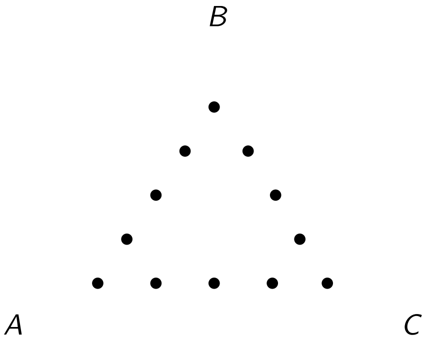
An ordered pair A, B of elements in $\mathrm{End}(V)$ is called **Lowering-Raising** (or **LR**) whenever there exists a decomposition of V that is lowered by A and raised by B.

It turns out that the decomposition is unique if it exists.

Definition (Ter 2015)

A 3-tuple of elements A, B, C in $\mathrm{End}(V)$ is called an **LR triple** whenever any two of A, B, C form an LR pair on V.

For $d = 4$ an LR triple A, B, C looks as follows:



"A, B, C pull toward their corner"

In [Ter LRT] we showed how to normalize an LR triple, and we classified up to isomorphism the normalized LR triples.

There are nine families of solutions, denoted

$$\mathrm{NBWeyl}^+_d(\mathbb{F}; j, q), \quad \mathrm{NBWeyl}^-_d(\mathbb{F}; j, q), \quad \mathrm{NBWeyl}^-_d(\mathbb{F}; t),$$
$$\mathrm{NBG}_d(\mathbb{F}; q), \quad \mathrm{NBG}_d(\mathbb{F}; 1),$$
$$\mathrm{NBNG}_d(\mathbb{F}; t),$$
$$\mathrm{B}_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0), \quad \mathrm{B}_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0), \quad \mathrm{B}_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0).$$

The last three families have a property called **bipartite**.

The relations for an LR triple

For each solution the elements A, B, C satisfy the following relations:

$\mathrm{NBWeyl}_d^{\pm}(\mathbb{F}; j, q)$:

$$qAB - q^{-1}BA = \vartheta_j I,$$
$$qBC - q^{-1}CB = \vartheta_j I,$$
$$qCA - q^{-1}AC = \vartheta_j I,$$

where

$$\vartheta_j = q^{-2j-1}(1 + q^{2j+1})^2(q - q^{-1})^{-1}.$$

The relations for an LR triple, cont.

$\mathrm{NBWeyl}_d^-(\mathbb{F}; t)$:

$$AB - tBA = \frac{2t}{1-t} I,$$
$$BC - tCB = \frac{2t}{1-t} I,$$
$$CA - tAC = \frac{2t}{1-t} I.$$

The relations for an LR triple, cont.

$\mathrm{NBG}_d(\mathbb{F}; q)$: A, B, C satisfy the $\mathbb{Z}_3$-symmetric down-up relations with parameters

$$\alpha = q^{-2}(q+1), \qquad \beta = -q^{-3}, \qquad \gamma = q^{-2}(q+1).$$

The relations for an LR triple, cont.

$\mathrm{NBG}_d(\mathbb{F}; 1)$: A, B, C satisfy the $\mathbb{Z}_3$-symmetric down-up relations with parameters

$$\alpha = 2, \qquad \beta = -1, \qquad \gamma = 2.$$

$\mathrm{NBNG}_d(\mathbb{F}; t)$: A, B, C satisfy the $\mathbb{Z}_3$-symmetric down-up relations with parameters

$$\alpha = 0, \qquad \beta = t^{-1}, \qquad \gamma = t^{-1} - 1.$$

The relations for an LR triple, cont.

$\mathrm{B}_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$:

$$A^3B + A^2BA - tABA^2 - tBA^3 = (\rho_0 + t/\rho_0)A^2,$$
$$B^3C + B^2CB - tBCB^2 - tCB^3 = (\rho'_0 + t/\rho'_0)B^2,$$
$$C^3A + C^2AC - tCAC^2 - tAC^3 = (\rho''_0 + t/\rho''_0)C^2$$

and also

$$AB^3 + BAB^2 - tB^2AB - tB^3A = (\rho_0 + t/\rho_0)B^2,$$
$$BC^3 + CBC^2 - tC^2BC - tC^3B = (\rho'_0 + t/\rho'_0)C^2,$$
$$CA^3 + ACA^2 - tA^2CA - tA^3C = (\rho''_0 + t/\rho''_0)A^2.$$

The relations for an LR triple, cont.

$\mathrm{B}_d(\mathbb{F}; 1, \rho_0, \rho_0', \rho_0'')$:

$$A^3B + A^2BA - ABA^2 - BA^3 = (\rho_0 + 1/\rho_0)A^2,$$
$$B^3C + B^2CB - BCB^2 - CB^3 = (\rho_0' + 1/\rho_0')B^2,$$
$$C^3A + C^2AC - CAC^2 - AC^3 = (\rho_0'' + 1/\rho_0'')C^2$$

and also

$$AB^3 + BAB^2 - B^2AB - B^3A = (\rho_0 + 1/\rho_0)B^2,$$
$$BC^3 + CBC^2 - C^2BC - C^3B = (\rho_0' + 1/\rho_0')C^2,$$
$$CA^3 + ACA^2 - A^2CA - A^3C = (\rho_0'' + 1/\rho_0'')A^2.$$

The relations for an LR triple, cont.

$\mathrm{B}_2(\mathbb{F}; \rho_0, \rho_0', \rho_0'')$: Same as $\mathrm{B}_d(\mathbb{F}; 1, \rho_0, \rho_0', \rho_0'')$.

In summary, we have the following result.

Theorem (Ter 2015)

Let A, B, C denote a nonbipartite LR triple over $\mathbb{F}$. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}$ such that A, B, C satisfy the $\mathbb{Z}_3$-symmetric down-up relations with parameters α, β, γ.

The $\mathbb{Z}_3$-symmetric down-up algebra $\mathbb{A}$ is related to the following familiar algebras:

the Weyl algebra,
the Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$,
the $\mathfrak{sl}_3$ loop algebra,
the Kac-Moody Lie algebra $A_2^{(1)}$,
the q-Weyl algebra,
the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$,
the quantized enveloping algebra $U_q(A_2^{(1)})$.

How $\mathbb{A}$ is related the Weyl algebra

Definition

For $\theta \in \mathbb{C}$, the **Weyl algebra** $W = W(\theta)$ is defined by generators A, B and the relation $AB - BA = \theta$.

Lemma

For the Weyl algebra $W(\theta)$, define $C = -A - B$. Then

$$AB - BA = \theta, \qquad BC - CB = \theta, \qquad CA - AC = \theta.$$

Theorem (Ter 2023)

Pick $\theta, \xi \in \mathbb{C}$ and consider the algebra $\mathbb{A} = \mathbb{A}(\xi + 1, -\xi, (\xi - 1)\theta)$. There exists an algebra homomorphism $\mathbb{A} \to W(\theta)$ that sends

$$A \mapsto A, \qquad B \mapsto B, \qquad C \mapsto C.$$

How $\mathbb{A}$ is related to $\mathfrak{sl}_2$.

For $n \geq 2$ the Lie algebra $\mathfrak{sl}_n$ consists of the matrices in $\mathrm{Mat}_n(\mathbb{C})$ that have trace 0, together with the Lie bracket $[x, y] = xy - yx$. The Lie algebra $\mathfrak{sl}_2$ has a basis

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Moreover

$$[A, B] = C - A - B, \quad [B, C] = A - B - C, \quad [C, A] = B - C - A.$$

Theorem (Ter 2023)

There exists an algebra homomorphism $\mathbb{A}(2, -1, 2) \to U(\mathfrak{sl}_2)$ that sends

$$A \mapsto A, \qquad B \mapsto B, \qquad C \mapsto C.$$

How $\mathbb{A}$ is related to $\mathfrak{sl}_3$.

Next, we describe how $\mathbb{A}$ is related to the Lie algebra $\mathfrak{sl}_3$.

Definition

Let $\xi \in \mathbb{C}$. We define elements A, B, C in $\mathfrak{sl}_3$ as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Theorem (Ter 2023)

There exists an algebra homomorphism $\mathbb{A}(2, -1, -2\xi) \to U(\mathfrak{sl}_3)$ *that sends*

$$A \mapsto A, \qquad B \mapsto B, \qquad C \mapsto C.$$

How $\mathbb{A}$ is related to the $\mathfrak{sl}_3$ loop algebra

Next, we describe how $\mathbb{A}$ is related to the loop algebra $\mathfrak{sl}_3 \otimes \mathbb{C}[t, t^{-1}]$

Definition

For $\xi \in \mathbb{C}$ we define elements A, B, C in $\mathfrak{sl}_3 \otimes \mathbb{C}[t, t^{-1}]$ as follows:

$$A = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & \xi t^{-1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \xi t^{-1} \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ \xi t^{-1} & 0 & 0 \\ t & 0 & 0 \end{pmatrix}.$$

Theorem (Ter 2023)

There exists an algebra homomorphism $\mathbb{A}(2, -1, -2\xi) \to U\big(\mathfrak{sl}_3 \otimes \mathbb{C}[t, t^{-1}]\big)$ *that sends*

$$A \mapsto A, \qquad B \mapsto B, \qquad C \mapsto C.$$

Next, we describe how $\mathbb{A}$ is related to the Kac-Moody Lie algebra $A_2^{(1)}$.

Consider the Cartan matrix $\mathcal{C}$ of type $A_2^{(1)}$:

$$\mathcal{C} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Definition

Define the Lie algebra $A_2^{(1)}$ by generators

$$e_i, \quad f_i, \quad h_i \qquad\qquad (1 \leq i \leq 3)$$

and the following relations. For $1 \leq i, j \leq 3$,

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = \mathcal{C}_{i,j} e_j,$$
$$[h_i, f_j] = -\mathcal{C}_{i,j} f_j, \qquad [e_i, f_j] = \delta_{i,j} h_i,$$
$$[e_i, [e_i, e_j]] = 0, \qquad [f_i, [f_i, f_j]] = 0 \qquad (i \neq j).$$

We call $A_2^{(1)}$ the **Kac-Moody Lie algebra** with Cartan matrix $\mathcal{C}$.

How $\mathbb{A}$ is related to the Kac-Moody Lie algebra $A_2^{(1)}$

Theorem (Ter 2023)

For $\xi \in \mathbb{C}$, there exists an algebra homomorphism $\mathbb{A}(2, -1, -2\xi) \to U(A_2^{(1)})$ that sends

$$A \mapsto e_1 + \xi f_2, \qquad B \mapsto e_2 + \xi f_3, \qquad C \mapsto e_3 + \xi f_1.$$

How $\mathbb{A}$ is related to $U_q(A_2^{(1)})$

Fix a nonzero $q \in \mathbb{C}$ such that $q^2 \neq 1$.

Definition

Define the algebra $U_q(A_2^{(1)})$ by generators

$$E_i, \quad F_i, \quad K_i, \quad K_i^{-1} \qquad\qquad (1 \leq i \leq 3)$$

and the following relations. For $1 \leq i, j \leq 3$,

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i K_j = K_j K_i,$$
$$K_i E_j K_i^{-1} = q^{\mathcal{C}_{i,j}} E_j, \qquad K_i F_j K_i^{-1} = q^{-\mathcal{C}_{i,j}} F_j,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \qquad (i \neq j),$$
$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \qquad (i \neq j).$$

How $\mathbb{A}$ is related to $U_q(A_2^{(1)})$

We call $U_q(A_2^{(1)})$ the **quantized enveloping algebra** of $A_2^{(1)}$.

It turns out that $K = K_1 K_2 K_3$ is central in $U_q(A_2^{(1)})$.

Theorem (Ter 2023)

Let $\xi \in \mathbb{C}$ and define

$$\alpha = q^3(q+q^{-1}), \qquad \beta = -q^6, \qquad \gamma = -\xi q^3(q+q^{-1}).$$

Then there exists an algebra homomorphism $\mathbb{A}(\alpha,\beta,\gamma) \to U_q(A_2^{(1)})$ that sends

$$A \mapsto \left(E_1 + \xi F_2 K^{-1}\right) K_3, \qquad B \mapsto \left(E_2 + \xi F_3 K^{-1}\right) K_1,$$
$$C \mapsto \left(E_3 + \xi F_1 K^{-1}\right) K_2.$$

Summary

In this talk, we first reviewed the **down-up algebra** $\mathcal{A}$ and its application to posets.

We then introduced the $\mathbb{Z}_3$**-symmetric down-up algebra** $\mathbb{A}$, and explained its connection to LR triples.

We then described how $\mathbb{A}$ is related to many familiar algebras in the literature.

THANK YOU FOR YOUR ATTENTION!