The Norton-balanced condition for Q-polynomial distance-regular graphs

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Overview

In this talk, we consider a **distance-regular graph** Γ.

We first review how a Q-polynomial primitive idempotent E of Γ satisfies the **balanced set condition**.

We then introduce a variation of the balanced set condition, called the **Norton-balanced condition**.

This condition involves the **Norton algebra** associated with E.

We list many examples that satisfy the Norton-balanced condition.

We also give some theoretical results.

Preliminaries

Throughout this talk, Γ denotes a distance-regular graph with vertex set X, path-length distance function ∂, and diameter $D \geq 3$.

For $x \in X$ and $0 \leq i \leq D$, define the set

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

Preliminaries, cont.

Let V denote the vector space over $\mathbb{R}$, consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{R}$.

The vector space V becomes a Euclidean space with bilinear form $\langle u, v \rangle = u^t v$ for $u, v \in V$.

We call V the **standard module**.

Preliminaries, cont.

For a vertex $x \in X$ define a vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{x} | x \in X\}$ form an orthonormal basis for V.

The adjacency matrix A of Γ acts on V.

Primitive idempotents

For an eigenvalue θ of A, the corresponding **primitive idempotent** E acts as the identity on the θ-eigenspace, and as zero on every other eigenspace of A.

The θ-eigenspace is equal to EV.

The subspace EV is spanned by the vectors $\{E\hat{x} | x \in X\}$.

We consider the linear dependencies among the vectors $\{E\hat{x} | x \in X\}$.

The Q-polynomial property

Definition (Ter 1987)

The primitive idempotent E is called **Q-polynomial** whenever the following (i), (ii) hold:

(i) the vectors $\{E\hat{x} | x \in X\}$ are mutually distinct;

(ii) for $x, y \in X$ and $0 \leq i, j \leq D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z} \in \mathrm{Span}\{E\hat{x} - E\hat{y}\}.$$

The above condition (ii) is called the **balanced set condition**.

Comments on the balanced set condition

For the rest of this talk, we assume that the primitive idempotent E is Q-polynomial.

Next, we mention a special case of the balanced set dependency.

Comments on the balanced set condition, cont.

Pick vertices $x, y \in X$ and write $i = \partial(x, y)$. Define

$$x_y^- = \sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} \hat{z},$$

$$x_y^+ = \sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} \hat{z},$$

where $\Gamma_{-1}(x) = \emptyset$ and $\Gamma_{D+1}(x) = \emptyset$.

Comments on the balanced set condition, cont.

By the balanced set condition,

$$Ex_y^- - Ey_x^- \in \mathrm{Span}\{E\hat{x} - E\hat{y}\},$$
$$Ex_y^+ - Ey_x^+ \in \mathrm{Span}\{E\hat{x} - E\hat{y}\}.$$

The symmetric balanced set dependency

We have been discussing the balanced set dependencies for the vectors $\{E\hat{x} | x \in X\}$.

These vectors satisfy another type of dependency, called the **symmetric balanced set dependency**.

This type of dependency is explained on the next slide.

The symmetric balanced set dependency, cont.

Lemma (Ter 1995)

For $x, y \in X$ and $0 \le i, j \le D$ we have

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} + \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z}$$

$$\in \mathrm{Span}\{Ex_y^- + Ey_x^-, Ex_y^+ + Ey_x^+, E\hat{x} + E\hat{y}\}.$$

The symmetric balanced set dependency, cont.

Combining the balanced set dependency and its symmetric version, we obtain the following result.

Lemma (Ter 1995)

For $x, y \in X$ and $0 \leq i, j \leq D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \mathrm{Span}\{Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}\}.$$

The strongly balanced condition

It could happen that for all $x, y \in X$ the following vectors are linearly dependent:

$$Ex_y^-, \quad Ex_y^+, \quad E\hat{x}, \quad E\hat{y}.$$

We now consider some situations where this occurs.

The strongly balanced condition, cont.

Definition (Ter 1988)

The set of vectors $\{E\hat{x} | x \in X\}$ is called **strongly balanced** whenever for all $x, y \in X$ and $0 \leq i, j \leq D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \mathrm{Span}\{E\hat{x}, E\hat{y}\}.$$

Lemma (Ter 1988)

The following are equivalent:

(i) *the set $\{E\hat{x} | x \in X\}$ is strongly balanced;*

(ii) *E is dual-bipartite or almost dual-bipartite.*

E being **dual-bipartite** means that the Krein parameter $a^*_i = 0$ for $0 \leq i \leq D$.

E being **almost dual-bipartite** means that $a^*_i = 0$ for $0 \leq i \leq D - 1$ and $a^*_D \neq 0$.

Next, we recall the **Norton algebra** structure on EV.

We will use the following notation.

The Norton algebra, cont.

For $u \in V$ and $x \in X$ let u_x denote the x-coordinate of u.

So

$$u = \sum_{x \in X} u_x \hat{x}.$$

For $u, v \in V$ define a vector

$$u \circ v = \sum_{x \in X} u_x v_x \hat{x}.$$

The Norton algebra, cont.

Definition (Cameron, Goethals, Seidel 1978)

The **Norton algebra** consists of the $\mathbb{R}$-vector space EV, together with the product

$$u \star v = E(u \circ v) \qquad\qquad (u, v \in EV).$$

The Norton product $\star$ is commutative, and nonassociative in general.

The Norton-balanced condition

We now introduce the Norton-balanced condition.

Definition (Ter 2024)

The set of vectors $\{E\hat{x} | x \in X\}$ is called **Norton-balanced** whenever for all $x, y \in X$ and $0 \le i, j \le D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

The Norton-balanced condition, cont.

Let us clarify the Norton-balanced condition.

Lemma (Ter 2024)

For the primitive idempotent E the following are equivalent:

(i) *the set $\{E\hat{x} | x \in X\}$ is Norton-balanced;*

(ii) *for all $x, y \in X$ we have*

$$Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

Some definitions

Definition

We say that Γ is Q-**polynomial** whenever Γ has at least one Q-polynomial primitive idempotent.

Definition

We say that Γ is **Norton-balanced** whenever Γ has a Q-polynomial primitive idempotent E such that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.

Next, we describe our results.

We have two kinds of results; some are about examples, and some are more theoretical.

We first describe the results about examples.

This will be done over the next few slides.

Norton-balanced examples

Assume that Γ is Q-polynomial. Using elementary arguments, we showed that Γ is Norton-balanced in the following cases:

(i) Γ is bipartite;

(ii) Γ is almost bipartite;

(iii) Γ is dual-bipartite;

(iv) Γ is almost dual-bipartite;

(v) Γ is tight.

Γ being **tight** means that Γ is not bipartite and $a_D = 0$.

More Norton-balanced examples

The 2021 book **Algebraic Combinatorics** by Bannai, Bannai, Ito, Tanaka gives a list of the known infinite families of Q-polynomial distance-regular graphs with unbounded diameter.

For each listed graph, every Q-polynomial structure is described.

We examined these Q-polynomial structures.

For each listed graph Γ and each Q-polynomial primitive idempotent E of Γ, we determined if the set $\{E\hat{x} | x \in X\}$ is Norton-balanced or not.

More Norton-balanced examples, cont.

In summary form, our conclusion is that Γ is Norton-balanced in the following cases:

(vi) Γ is a Hamming graph;

(vii) Γ is a Johnson graph;

(viii) Γ is the Grassmann graph $J_q(2D, D)$;

(ix) Γ is a halved bipartite dual-polar graph;

(x) Γ is a halved Hemmeter graph;

(xi) Γ is a halved hypercube;

(xii) Γ is a folded-half hypercube.

Another Norton-balanced example

The Norton-balanced condition was inspired by our recent work with Kazumasa Nomura on **spin models**.

We show that Γ is Norton-balanced in the following case:

(xiii) Γ has q-Racah type and affords a spin model.

The nature of the Norton-balanced condition

We show that in general, Γ being Norton-balanced is not a condition on the intersection numbers alone.

To show this, we consider the Hamming graph $H(D, 4)$ and a Doob graph with diameter D.

These graphs have the same intersection numbers.

We showed that $H(D, 4)$ is Norton-balanced and the Doob graph is not.

In a moment, we will describe our theoretical results.

We will use the following definition.

The reinforced condition

Definition

We say that Γ is **reinforced** whenever the following (i), (ii) hold for $2 \leq i \leq D$:

(i) for $x, y \in X$ at distance $\partial(x, y) = i$, the average valency of the induced subgraph $\Gamma(x) \cap \Gamma_{i-1}(y)$ is independent of x and y;

(ii) for $x, y \in X$ at distance $\partial(x, y) = i - 1$, the average valency of the induced subgraph $\Gamma(x) \cap \Gamma_i(y)$ is independent of x and y.

If Γ is distance-transitive then Γ is reinforced.

The reinforced condition, cont.

Assume for the moment that Γ is reinforced.

For $2 \leq i \leq D$ let z_i denote the average valency mentioned in (i), and note that $a_1 - z_i$ is the average valency mentioned in (ii).

The scalar z_i is often called the ith **kite number**.

It is known that the kite numbers $\{z_i\}_{i=2}^{D}$ are determined by z_2 and the intersection numbers of Γ.

Theoretical results

We now summarize our theoretical results.

This will be done over the next few slides.

Let E denote a Q-polynomial primitive idempotent of Γ.

Theoretical results, cont.

Consider the following two conditions on E:

(i) the set $\{E\hat{x} | x \in X\}$ is Norton-balanced;

(ii) for $x, y \in X$ the vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent.

By our earlier comments, (i) implies (ii).

Using an example (the Hermitean forms graph with $q = -2$) we showed that (ii) does not imply (i).

We showed that (i) is implied by (ii) together with a certain restriction on the coefficients in the linear dependence.

The polynomials $\Phi_i(\lambda)$

Let λ denote an indeterminate.

For $2 \leq i \leq D-1$ we define a quadratic polynomial $\Phi_i(\lambda)$ whose coefficients are determined by the intersection numbers of Γ.

The polynomial $\Phi_i(\lambda)$ has the following meaning.

Pick $x, y \in X$ at distance $\partial(x, y) = i$.

The polynomials $\Phi_i(\lambda)$, cont.

Assuming that Γ is reinforced, we compute the inner products between Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ in terms of the intersection numbers and z_i, z_{i+1}.

Using these inner products and a Cauchy-Schwarz inequality, we show that $\Phi_i(z_2) \geq 0$, with equality iff Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent.

Consequently...

The polynomials $\Phi_i(\lambda)$, cont.

Lemma

Assume that Γ is reinforced and the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Then $\Phi_i(z_2) = 0$ for $2 \le i \le D-1$.

Definition

We say that E is a **dependency candidate** (or **DC**) whenever there exists $\xi \in \mathbb{C}$ such that $\Phi_i(\xi) = 0$ for $2 \le i \le D-1$.

Lemma

Assume that Γ is reinforced and the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Then E is DC.

The polynomials $\Phi_i(\lambda)$, cont.

Note that E being DC is a condition on the intersection numbers of Γ.

In our main theoretical result, we display a necessary and sufficient condition on the intersection numbers of Γ, for E to be DC.

This condition is shown on the next two slides.

The polynomials $\Phi_i(\lambda)$, cont.

Theorem (Ter 2024)

For $D \geq 4$ the following (i), (ii) hold.

(i) *Assume that the type of E is included in the table below. Then E is DC iff at least one of the listed scalars is zero.*

type of E	E is DC iff at least one of these scalars is 0
q-Racah	$a_1^*, \quad r_1^2 - s, \quad r_2^2 - s, \quad r_3^2 - s,$ $s + s^* - q^{-1}r_1 - q^{-1}r_2 + r_3 + r_1r_2 - qr_2r_3 - qr_3r_1,$ $s + s^* - q^{-1}r_2 - q^{-1}r_3 + r_1 + r_2r_3 - qr_3r_1 - qr_1r_2,$ $s + s^* - q^{-1}r_3 - q^{-1}r_1 + r_2 + r_3r_1 - qr_1r_2 - qr_2r_3$
q-Hahn	$a_1^*, \quad s^* - q^{-1}r + r_3 - qrr_3,$ $s^* - q^{-1}r_3 + r - qrr_3, \quad s^* - q^{-1}r_3 - q^{-1}r + rr_3$
dual q-Hahn	$a_1^*, \quad r^2 - s, \quad r_3^2 - s, \quad s - q^{-1}r + r_3 - qrr_3,$ $s - q^{-1}r_3 + r - qrr_3, \quad s - q^{-1}r_3 - q^{-1}r + rr_3$
affine q-Krawtchouk	$a_1^*, \quad -q^{-1}r + r_3 - qrr_3,$ $-q^{-1}r_3 + r - qrr_3, \quad -q^{-1}r_3 - q^{-1}r + rr_3$

The polynomials $\Phi_i(\lambda)$, cont.

Theorem

(i) *continued..*

type of E	E is DC iff at least one of these scalars is 0
dual q-Krawtchouk	$a_1^*, \quad r_3^2 - s, \quad s + r_3, \quad s - q^{-1}r_3$
Racah	$a_1^*, \quad 2r_1 - s, \quad 2r_2 - s, \quad 2r_3 - s,$ $2r_1r_2 - 2r_3 - 2 - ss^*, \quad 2r_2r_3 - 2r_1 - 2 - ss^*,$ $2r_3r_1 - 2r_2 - 2 - ss^*$
Hahn	$a_1^*, \quad 2r - s^*, \quad 2r_3 - s^*$

(ii) *Assume that the type of E is q-Krawtchouk or dual Hahn or Krawtchouk or Bannai/Ito. Then E is DC.*

We used the above theorem to show that certain distance-regular graphs are not Norton-balanced.

Summary

In this talk, we considered a **distance-regular graph** Γ.

We reviewed how a Q-polynomial primitive idempotent E of Γ satisfies the **balanced set condition**.

We then introduced a variation of the balanced set condition, called the **Norton-balanced condition**.

We listed many examples that satisfy the Norton-balanced condition.

We then introduced the closely related DC **condition** on E. We gave a necessary/sufficient condition on the intersection numbers of Γ, for E to be DC.

THANK YOU FOR YOUR ATTENTION!