The nucleus of a Q-polynomial distance-regular graph

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Overview

In this talk, we consider a Q-polynomial distance-regular graph Γ.

For a vertex x of Γ the corresponding **subconstituent algebra** $T = T(x)$ is generated by the adjacency matrix A and the dual adjacency matrix $A^* = A^*(x)$ with respect to x.

We introduce a T-module $\mathcal{N} = \mathcal{N}(x)$ called the **nucleus** of Γ with respect to x.

We will show that the irreducible T-submodules of $\mathcal{N}$ have a property called **thin**.

Under the assumption that Γ is a nonbipartite **dual polar graph**, we give an explicit basis for $\mathcal{N}$ and the action of A, A^* on this basis.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, with vertex set X and adjacency relation $\mathcal{R}$.

Let ∂ denote the path-length distance function for Γ, and recall the **diameter**

$$D = \max\{\partial(x, y) | x, y \in X\}.$$

For $x \in X$ and $0 \leq i \leq D$ define the set

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We call $\Gamma_i(x)$ the i**th subconstituent of** Γ **with respect to** x.

The graph Γ is called **distance-regular** whenever for all $0 \leq h, i, j \leq D$ and $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{i,j} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y.

The $p^h_{i,j}$ are called the **intersection numbers** of Γ.

For the rest of this talk, we assume that Γ is distance-regular with $D \geq 1$.

Distance-regular graphs, cont.

By construction $p^h_{i,j} = p^h_{j,i}$ for $0 \leq h, i, j \leq D$.

By the **triangle inequality**, the following hold for $0 \leq h, i, j \leq D$:

(i) $p^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

We abbreviate

$$c_i = p^i_{1,i-1} \ (1 \leq i \leq D), \qquad a_i = p^i_{1,i} \ (0 \leq i \leq D),$$
$$b_i = p^i_{1,i+1} \ (0 \leq i \leq D-1).$$

The distance matrices of Γ

We recall the distance matrices of Γ.

Let the algebra $\mathrm{Mat}_X(\mathbb{C})$ consist of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

For $0 \le i \le D$ define $A_i \in \mathrm{Mat}_X(\mathbb{C})$ that has (y,z)-entry

$$(A_i)_{y,z} = \begin{cases} 1, & \text{if } \partial(y,z) = i; \\ 0, & \text{if } \partial(y,z) \neq i \end{cases} \qquad (y, z \in X).$$

We call A_i the ith **distance matrix** of Γ. We call $A = A_1$ the **adjacency matrix** of Γ.

The Bose-Mesner algebra of Γ

For $0 \leq i, j \leq D$ we have

$$A_i A_j = \sum_{h=0}^{D} p^h_{i,j} A_h.$$

Consequently the matrices $\{A_i\}_{i=0}^{D}$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, called the **Bose-Mesner algebra** of Γ.

It turns out that A generates M.

The primitive idempotents of Γ

The matrices $\{A_i\}_{i=0}^D$ are symmetric and mutually commute, so they can be simultaneously diagonalized over the real numbers.

Consequently M has a second basis $\{E_i\}_{i=0}^D$ such that

(i) $E_0 = |X|^{-1}J$ (J has all entries 1)

(ii) $I = \sum_{i=0}^D E_i$;

(iii) $E_iE_j = \delta_{i,j}E_i$ $(0 \le i, j \le D)$.

We call $\{E_i\}_{i=0}^D$ the **primitive idempotents** of Γ.

We recall the Krein parameters of Γ.

The Bose-Mesner algebra M is closed under the **entry-wise product** $\circ$, because $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq D$.

Consequently, there exist scalars $q^h_{i,j} \in \mathbb{C}$ $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{i,j} E_h \qquad (0 \leq i, j \leq D).$$

The scalars $q^h_{i,j}$ are called the **Krein parameters** of Γ.

By construction $q^h_{i,j} = q^h_{j,i}$ $(0 \leq h, i, j \leq D)$.

The Q-polynomial property

The ordering $\{E_i\}_{i=0}^D$ is said to be Q-**polynomial** whenever the following hold for $0 \leq h, i, j \leq D$:

(i) $q^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $q^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

For the rest of this talk, we assume that the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial.

The dual Bose-Mesner algebra

We recall the dual Bose-Mesner algebras of Γ.

For the rest of this talk, fix a vertex $x \in X$. We call x the **base vertex**.

For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } y \in \Gamma_i(x); \\ 0, & \text{if } y \notin \Gamma_i(x) \end{cases} \qquad (y \in X).$$

We call E_i^* the ith **dual primitive idempotent of Γ with respect to** x.

The dual Bose-Mesner algebra, cont.

Note that

$$I = \sum_{i=0}^{D} E_i^*, \qquad E_i^* E_j^* = \delta_{i,j} E_i^* \quad (0 \le i, j \le D).$$

Consequently the matrices $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$.

We call M^* the **dual Bose-Mesner algebra of Γ with respect to** x.

The dual distance matrices

We recall the dual distance matrices of Γ.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \qquad (y \in X).$$

It turns out that $\{A_i^*\}_{i=0}^D$ form a basis for M^*. Moreover

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{i,j}^h A_h^* \qquad (0 \leq i, j \leq D).$$

We call A_i^* the ith **dual distance matrix of** Γ (with respect to x and the given Q-polynomial structure).

We call $A^* = A_1^*$ the **dual adjacency matrix of** Γ (with respect to x and the given Q-polynomial structure).

In turns out that A^* generates M^*.

The subconstituent algebra T

We recall the subconstituent algebra.

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

The algebra T is finite-dimensional and noncommutative.

We call T the **subconstituent algebra of Γ with respect to x**.

Note that T is generated by A, A^*.

The eigenvalues and dual eigenvalues

For $0 \leq i \leq D$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E^*_i).

We have

$$A = \sum_{i=0}^{D} \theta_i E_i, \qquad A^* = \sum_{i=0}^{D} \theta^*_i E^*_i.$$

The standard module V

Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$, consisting of the column vectors that have coordinates indexed by X and all entries in $\mathbb{C}$.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**.

A basis for V

We endow V with a Hermitean form $\langle \, , \, \rangle$ such that $\langle u, v \rangle = u^{\mathrm{t}} \overline{v}$ for all $u, v \in V$. Here t denotes transpose and $-$ denotes complex conjugation.

For $y \in X$ define a vector $\hat{y} \in V$ that has y-coordinate 1 and all other coordinates 0.

Observe that the vectors $\{\hat{y} | y \in X\}$ form an orthonormal basis for V.

For convenience, we adopt the following convention.

By a T-**module**, we mean a T-submodule of the standard module V.

Irreducible T-modules

Definition

A T-module W is said to be **irreducible** whenever $W \neq 0$ and W does not contain a T-module besides 0 and W.

Lemma (Ter 92)

Every T-module is an orthogonal direct sum of irreducible T-modules. In particular, the standard T-module V is an orthogonal direct sum of irreducible T-modules.

The thin condition

Definition

Let W denote an irreducible T-module. It is known that the following are equivalent:

(i) $\dim E_i W \leq 1$ $(0 \leq i \leq D)$;

(ii) $\dim E_i^* W \leq 1$ $(0 \leq i \leq D)$.

We say that W is **thin** whenever (i), (ii) hold.

Let W denote an irreducible T-module. By the **endpoint** of W we mean

$$\min\{i|0 \leq i \leq D,\ E_i^* W \neq 0\}.$$

By the **dual endpoint** of W, we mean

$$\min\{i|0 \leq i \leq D,\ E_i W \neq 0\}.$$

By the **diameter** of W, we mean

$$\left|\{i|0 \leq i \leq D,\ E_i^* W \neq 0\}\right| - 1.$$

By [Pascasio 2003] the diameter of W is equal to

$$\left|\{i|0 \leq i \leq D,\ E_i W \neq 0\}\right| - 1.$$

The primary T-module

Example (Ter 92)

There exists a unique irreducible T-module that has diameter D; this T-module is called **primary**. An irreducible T-module is primary iff it has endpoint 0 iff it has dual endpoint 0. The primary T-module is thin.

The Caughman bound

Lemma

Let W denote an irreducible T-module, with endpoint r and diameter d. Then the following hold.

(i) *[Caughman 99]* $2r - D + d \geq 0$.

(ii) *If equality holds in* (i) *then W is thin.*

The dual Caughman bound

Lemma

Let W denote an irreducible T-module, with dual endpoint t and diameter d. Then the following hold.

(i) *[Caughman 99]* $2t - D + d \geq 0$.

(ii) *If equality holds in* (i) *then W is thin.*

An inequality

Next, we combine the Caughman bound and the dual Caughman bound into one inequality.

Theorem

Let W denote an irreducible T-module, with endpoint r, dual endpoint t, and diameter d. Then

$$r + t - D + d \geq 0.$$

Moreover, equality holds iff both $t = r$ and $d = D - 2r$.

To prove the theorem, note that

$$r + t - D + d = \frac{2r - D + d}{2} + \frac{2t - D + d}{2}.$$

The displacement

Motivated by the previous inequality, we make a definition.

Definition (Ter 2005)

Let W denote an irreducible T-module. By the **displacement** of W, we mean the integer

$$r + t - D + d,$$

where r (resp. t) (resp. d) denotes the endpoint (resp. dual endpoint) (resp. diameter) of W.

Example

The primary T-module has displacement 0.

The definition of the nucleus

We are now ready to define the nucleus.

Definition (Ter 24)

By the **nucleus** of Γ with respect to x, we mean the span of the irreducible T-modules that have displacement 0.

By construction, the nucleus of Γ with respect to x is a T-module that contains the primary irreducible T-module with respect to x.

In the next slide, we emphasize a few more points about the nucleus.

Describing the nucleus, cont.

Lemma

Let W denote an irreducible T-submodule of the nucleus, with endpoint r, dual endpoint t, and diameter d. Then:

(i) $0 \leq r \leq D/2$;

(ii) $t = r$;

(iii) $d = D - 2r$;

(iv) W *is thin.*

The nucleus from another point of view

So far, we used the concept of displacement to define a T-module called the nucleus.

Next, we describe the nucleus from another point of view.

An intersection

Lemma (Ter 2005)

For $0 \leq i, j \leq D$ such that $i + j < D$,

$$(E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V) = 0.$$

The subspaces $\mathcal{N}_i$

Definition

For $0 \leq i \leq D$ define a subspace $\mathcal{N}_i = \mathcal{N}_i(x)$ by

$$\mathcal{N}_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{D-i} V).$$

The subspaces $\mathcal{N}_i$, cont.

Lemma (Ter 2005)

The sum $\sum_{i=0}^{D} \mathcal{N}_i$ is direct.

The subpace $\mathcal{N}$

Definition

Define a subspace $\mathcal{N} = \mathcal{N}(x)$ by

$$\mathcal{N} = \sum_{i=0}^{D} \mathcal{N}_i.$$

The nucleus, revisited

Theorem (Ter 2024)

The following are the same:

(i) *the subspace* $\mathcal{N} = \mathcal{N}(x)$*;*

(ii) *the nucleus of* Γ *with respect to* x*.*

For the rest of this talk, we consider a family of Q-polynomial distance-regular graphs called the **dual polar graphs**.

These graphs are defined on the next four slides.

The dual polar graphs

Example

Let $\mathbf{U}$ denote a finite vector space with one of the following nondegenerate forms:

name	dim($\mathbf{U}$)	field	form	e
$B_D(p^n)$	$2D+1$	$GF(p^n)$	quadratic	0
$C_D(p^n)$	$2D$	$GF(p^n)$	symplectic	0
$D_D(p^n)$	$2D$	$GF(p^n)$	quadratic (Witt index D)	-1
$^2D_{D+1}(p^n)$	$2D+2$	$GF(p^n)$	quadratic (Witt index D)	1
$^2A_{2D}(p^n)$	$2D+1$	$GF(p^{2n})$	Hermitean	$1/2$
$^2A_{2D-1}(p^n)$	$2D$	$GF(p^{2n})$	Hermitean	$-1/2$

The dual polar graphs, cont.

Example (continued...)

A subspace of $\mathbf{U}$ is called **isotropic** whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is D. The corresponding dual polar graph Γ has vertex set X consisting of the maximal isotropic subspaces of $\mathbf{U}$. Vertices $y, z \in X$ are adjacent whenever $y \cap z$ has dimension $D - 1$. More generally, $\partial(y, z) = D - \dim y \cap z$.

Example (continued..)

The graph Γ is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^i - 1}{q - 1}, \quad a_i = (q^{e+1} - 1)\frac{q^i - 1}{q - 1}, \quad b_i = q^{e+1}\frac{q^D - q^i}{q - 1}$$

for $0 \leq i \leq D$, where $q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n}$.

The dual polar graphs are Q-polynomial

Example (continued..)

The graph Γ has a Q-polynomial structure such that

$$\theta_i = q^{e+1}\frac{q^D - 1}{q-1} - \frac{(q^i-1)(q^{D+e+1-i}+1)}{q-1} \qquad (0 \le i \le D),$$

$$\theta^*_i = \frac{q^{D+e}+q}{q^e+1}\,\frac{q^{-i}(q^{D+e}+1)-q^e-1}{q-1} \qquad (0 \le i \le D).$$

From now on, we assume that Γ is a dual polar graph that is nonbipartite ($e \neq -1$).

It is known that every irreducible T-module is thin.

The q-binomial coefficients

We bring in some notation. For an integer $n \geq 0$ define

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

We further define

$$[n]^!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]^!_q = 1$. For $0 \leq i \leq n$ define the q-binomial coefficient

$$\binom{n}{i}_q = \frac{[n]^!_q}{[i]^!_q [n-i]^!_q}.$$

Our next goal is to find an orthogonal basis for the nucleus $\mathcal{N} = \mathcal{N}(x)$.

The relation $\sim$

Definition

Using the vertex x, we define a binary relation $\sim$ on X as follows. For $y, z \in X$ we declare $y \sim z$ whenever both

(i) $\partial(x, y) = \partial(x, z)$;

(ii) y, z are in the same connected component of $\Gamma_i(x)$, where $i = \partial(x, y) = \partial(x, z)$.

Note that $\sim$ is an equivalence relation.

An orthogonal basis for the nucleus

Theorem (Ter 2024)

The nucleus $\mathcal{N}$ has an orthogonal basis consisting of the characteristic vectors of the $\sim$ equivalence classes.

The projective geometry $L_D(q)$

In order to describe $\mathcal{N}$ in more detail, we bring in a projective geometry.

In what follows, we work with the finite field $GF(q)$ associated with Γ from the definition of a dual polar graph.

The projective geometry $L_D(q)$

Definition

Let $\mathbf{V}$ denote a vector space over $GF(q)$ that has dimension D. Let the set $\mathcal{P}$ consist of the subspaces of $\mathbf{V}$. Define a partial order $\leq$ on $\mathcal{P}$ such that for $\eta, \zeta \in \mathcal{P}$, $\eta \leq \zeta$ whenever $\eta \subseteq \zeta$. The poset $\mathcal{P}, \leq$ is denoted $L_D(q)$ and called a **projective geometry**.

The projective geometry $L_D(q)$

Recall our fixed vertex $x \in X$.

By the definition of a dual polar graph, the vertex x is a vector space over $GF(q)$ that has dimension D.

For notational convenience, we always take the $\mathbf{V} = x$.

Definition

For $\eta, \zeta \in \mathcal{P}$, we say that ζ **covers** η whenever $\eta \subseteq \zeta$ and $\dim \zeta - \dim \eta = 1$. We say that η, ζ are **adjacent** whenever one of η, ζ covers the other one. The set $\mathcal{P}$ together with the adjacency relation, forms an undirected graph. For $\eta \in \mathcal{P}$, let the set $\mathcal{P}(\eta)$ consist of the elements in $\mathcal{P}$ that are adjacent to η. For $0 \leq i \leq D$, let the set $\mathcal{P}_i$ consist of the elements in $\mathcal{P}$ that have dimension $D - i$. Note that $\mathcal{P}_0 = \{x\}$. For notational convenience, define $\mathcal{P}_{-1} = \emptyset$ and $\mathcal{P}_{D+1} = \emptyset$.

In the next slide, we describe some basic combinatorial features of $\mathcal{P}$.

Lemma

For $0 \leq i \leq D$, each vertex in $\mathcal{P}_i$ is adjacent to exactly $[i]_q$ vertices in $\mathcal{P}_{i-1}$ and exactly $[D-i]_q$ vertices in $\mathcal{P}_{i+1}$.

Lemma

We have

$$|\mathcal{P}_i| = \binom{D}{i}_q \qquad (0 \leq i \leq D).$$

Using $L_D(q)$ to describe $\mathcal{N}$

We have been discussing the set $\mathcal{P}$.

Earlier we found an orthogonal basis for the nucleus $\mathcal{N}$.

Our next goal is to display a bijection from $\mathcal{P}$ to this basis.

Using $L_D(q)$ to describe the relation $\sim$

The result below follows from the work of Chih-wen Weng concerning weak geodetically closed subgraphs (1998).

Lemma

For $y, z \in X$ the following are equivalent:

(i) $y \sim z$;

(ii) $x \cap y = x \cap z$.

Recall the standard module V of Γ.

Definition

For $\eta \in \mathcal{P}$ we define a vector $\eta^{\mathcal{N}} \in V$ as follows:

$$\eta^{\mathcal{N}} = \sum_{\substack{y \in X \\ x \cap y = \eta}} \hat{y}.$$

By construction, the above vector $\eta^{\mathcal{N}}$ is the characteristic vector of a $\sim$ equivalence class.

A bijection

Theorem (Ter 2024)

We give a bijection from $\mathcal{P}$ to our basis for $\mathcal{N}$. The bijection sends $\eta \to \eta^{\mathcal{N}}$ for all $\eta \in \mathcal{P}$.

Corollary

We have

$$\dim \mathcal{N} = |\mathcal{P}| = \sum_{i=0}^{D} \binom{D}{i}_q .$$

The action of A, A^* on the nucleus

We now bring in the adjacency matrix A of Γ, and the dual adjacency matrix $A^* = A^*(x)$ of Γ with respect to x.

Theorem (Ter 2024)

We give the action of A, A^* on the basis $\{\eta^{\mathcal{N}} | \eta \in \mathcal{P}\}$ for $\mathcal{N}$. For $0 \le i \le D$ and $\eta \in \mathcal{P}_i$ we have

$$A\eta^{\mathcal{N}} = a_1 \frac{q^i - 1}{q - 1} \eta^{\mathcal{N}} + \sum_{\zeta \in \mathcal{P}(\eta) \cap \mathcal{P}_{i+1}} \zeta^{\mathcal{N}} + (a_1 + 1) q^{i-1} \sum_{\zeta \in \mathcal{P}(\eta) \cap \mathcal{P}_{i-1}} \zeta^{\mathcal{N}};$$

$$A^* \eta^{\mathcal{N}} = \theta^*_i \eta^{\mathcal{N}}.$$

The action of A, A^* on the nucleus

The previous theorem shows that the action of A on $\mathcal{N}$ becomes a **weighted adjacency map** for $L_D(q)$.

We would like to acknowledge that a similar weighted adjacency map for $L_D(q)$ showed up earlier in the work of Bernard, Crampé, and Vinet [2022] concerning the dual polar graph with symplectic type and q a prime.

In this talk, we considered a Q-polynomial distance-regular graph Γ with diameter $D \geq 1$.

For a vertex x of Γ we considered the subconstituent algebra $T = T(x)$ generated by A and $A^* = A^*(x)$.

We introduced a T-module $\mathcal{N} = \mathcal{N}(x)$ called the **nucleus** of Γ with respect to x.

We showed that the irreducible T-submodules of $\mathcal{N}$ are thin.

Under the assumption that Γ is a nonbipartite dual polar graph, we gave an explicit basis for $\mathcal{N}$ and the action of A, A^* on this basis.

THANK YOU FOR YOUR ATTENTION!