15 Breakthroughs in Algebraic Combinatorics

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Overview

In this survey talk, I will review 15 **MAJOR EVENTS** and **CONCEPTUAL BREAKTHROUGHS** by other people that had a large impact on my own work.

The list is in **CHRONOLOGICAL ORDER**.

I am the first to admit that the list is **VERY SUBJECTIVE**.

Also, there are **SO MANY BREAKTHROUGHS** that I probably forgot some.

1. The Onsager Lie Algebra (1944).

In 1944 the physical chemist **Lars Onsager** obtained the free energy of the two-dimensional Ising model in a zero magnetic field.

In that work an infinite-dimensional Lie algebra was introduced; now called the **Onsager Lie algebra** and denoted by O.

Onsager won the **Nobel Prize in Chemistry** in 1968.

The original definition of O

The Onsager Lie algebra was **originally defined** as follows.

Definition (Onsager 1944)

The Lie algebra O has a basis $\{A_k\}_{k\in\mathbb{Z}}$, $\{B_{k+1}\}_{k\in\mathbb{N}}$ such that for $k, \ell \in \mathbb{Z}$,

$$[A_k, A_\ell] = 2B_{k-\ell},$$
$$[B_k, A_\ell] = A_{k+\ell} - A_{\ell-k},$$
$$[B_k, B_\ell] = 0.$$

In the above lines $B_0 = 0$ and $B_k + B_{-k} = 0$ for $k \in \mathbb{Z}$.

Much later, the following attractive presentation was found.

Lemma (J. Perk, 1989)

The Lie algebra O has a presentation by generators A, B and relations

$$[A,[A,[A,B]]] = 4[A,B],$$
$$[B,[B,[B,A]]] = 4[B,A].$$

The above relations are called the **Dolan/Grady relations**.

The Dolan/Grady relations show up in **Algebraic Combinatorics** in connection with the **Hamming graph** $H(D, N)$.

The vertex set X of $H(D, N)$ consists of the D-tuples of elements taken from the set $\{1, 2, \ldots, N\}$.

Two vertices are adjacent whenever they differ in exactly one coordinate.

The graph $H(D, N)$ is **distance-regular** with **diameter** D.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the **adjacency matrix** for $H(D, N)$.

For a vertex $x \in X$, the corresponding **dual adjacency matrix** $A^* = A^*(x)$ is the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$A^*_{y,y} = (N-1)D - iN, \qquad\qquad i = \partial(x, y)$$

for all $y \in X$.

It is routine to check that

$$[A,[A,[A,A^*]]] = N^2[A,A^*],$$
$$[A^*,[A^*,[A^*,A]]] = N^2[A^*,A].$$

So, up to normalization the A, A^* satisfy the Dolan/Grady relations.

The q-**Onsager algebra** O_q is a q-deformation of the universal enveloping algebra $U(O)$.

It is defined as follows.

Fix $0 \neq q \in \mathbb{C}$ such that $q^4 \neq 1$.

Recall the q-commutator

$$[R, S]_q = qRS - q^{-1}SR.$$

The q-Onsager algebra

Definition

The q-**Onsager algebra** O_q is defined by generators A, B and relations

$$[A,[A,[A,B]_q]_{q^{-1}}] = (q^2-q^{-2})^2[B,A],$$
$$[B,[B,[B,A]_q]_{q^{-1}}] = (q^2-q^{-2})^2[A,B].$$

The above relations are called the q-**Dolan/Grady relations**

History

I first heard about the Onsager Lie algebra quite late, around 1997, from Anatol Kirillov at a Madison conference on special functions.

At the conference I spoke about my 1992 discovery of the **subconstituent algebra** T of a Q-**polynomial distance-regular graph**.

The generators A, A^* of T satisfy two relations that I called the **tridiagonal relations**.

After my talk, Kirillov pointed out that the tridiagonal relations include the Dolan/Grady relations as a special case, and the Dolan/Grady relations are the defining relations for the Onsager Lie algebra.

From then on, it was natural to use the names q-**Dolan/Grady relations** and q-**Onsager algebra** for the most general case.

2. The Krein condition (1950).

Recall the primitive idempotents $\{E_i\}_{i=0}^{D}$ of a **commutative association scheme** $\Gamma = (X, \{R_i\}_{i=0}^{D})$.

Definition

The **Krein parameters** $q^h_{i,j}$ $(0 \le h, i, j \le D)$ of Γ are defined by

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{i,j} E_h \qquad (0 \le i, j \le D),$$

where $\circ$ denotes entrywise multiplication.

The Krein condition, cont.

It is a remarkable fact that $q^h_{i,j}$ is **real and nonnegative** for $0 \leq h, i, j \leq D$.

This fact is called the **Krein condition**.

This condition originated in the 1950 work of the Soviet mathematician **Mark Grigorievich Krein** (1907–1989) concerning harmonic analysis on homogeneous spaces.

Krein won the **Wolf Prize** in 1982.

The Krein condition, cont.

The Krein condition has always fascinated me.

It is a deep open problem to **explain the combinatorial meaning of the Krein parameters**.

I have been thinking about this problem for my entire career.

3. The Q-polynomial property (1973).

The Q-polynomial property is about the Krein parameters of a commutative association scheme $\Gamma = (X, \{R_i\}_{i=0}^D)$.

This property was introduced by **Philippe Delsarte** in his 1973 thesis on **coding theory**.

The Q-polynomial property, cont.

Definition (Delsarte 1973)

An ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ is said to be **Q-polynomial** whenever the following hold for $0 \le h, i, j \le D$:

(i) $q_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $q_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

The Q-polynomial property, cont.

Most of the known infinite families of distance-regular graphs with unbounded diameter are Q-polynomial.

It is a deep open problem to **explain why this happens**.

This is another problem that I have been thinking about for my entire career.

4. The Norton algebra (1978).

The following influential results were obtained in the paper:

P. Cameron, J. Geothals, J. Seidel. **The Krein condition, spherical designs, Norton algebras, and permutation groups**. Indag. Math. 40 (1978).

The Norton algebra, cont.

Consider a commutative association scheme $\Gamma = (X, \{R_i\}_{i=0}^D)$ with primitive idempotents $\{E_i\}_{i=0}^D$.

Let V denote the $\mathbb{C}$-vector space consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{C}$.

We call V the **standard module** of Γ.

The Norton algebra, cont.

For $x \in X$ define the vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{x} | x \in X\}$ form a basis for V.

The Norton algebra, cont.

Theorem (Cameron, Geothals, Seidel 1978)

For $0 \leq h, i, j \leq D$ *we have*

$$\sum_{x \in X} E_h \hat{x} \otimes E_i \hat{x} \otimes E_j \hat{x} = 0$$

if and only if

$$q^h_{i,j} = 0.$$

The Norton algebra, cont.

The previous theorem is a bit obscure, so we give a reformulation.

Theorem (Cameron, Geothals, Seidel 1978)

The following hold for $0 \leq h, i, j \leq D$.

(i) *Assume that* $q^h_{i,j} \neq 0$. *Then* $E_h\big(E_iV \circ E_jV\big)$ *spans* E_hV.

(ii) *Assume that* $q^h_{i,j} = 0$. *Then* $E_h\big(E_iV \circ E_jV\big) = 0$.

Here $\circ$ *denotes entry-wise multiplication.*

Around 1985–1990 I struggled to understand the previous theorem.

The struggle eventually lead me to the subconstituent algebra in 1992.

In the subconstituent algebra we have the triple product relations

$$E^*_h A_i E^*_j = 0 \qquad \text{iff} \quad p^h_{i,j} = 0, \tag{1}$$

$$E_h A^*_i E_j = 0 \qquad \text{iff} \quad q^h_{i,j} = 0. \tag{2}$$

The triple product relations (2) are the same thing as the previous theorem in disguise.

The Norton algebra is defined as follows.

Definition

For the commutative association scheme $\Gamma = (X, \{R_i\}_{i=0}^{D})$ and $0 \leq j \leq D$ the jth **Norton algebra** of Γ consists of the $\mathbb{C}$-vector space $E_j V$ together with the multiplication

$$u \star v = E_j(u \circ v) \qquad (u, v \in E_j V).$$

The Norton algebra, cont.

The Norton algebra $E_j V$ is commutative, but nonassociative in general.

Of particular interest is the Norton algebra $E_1 V$ under the assumption that the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial.

It is a deep open problem to **describe the algebraic structure of this Norton algebra**.

5. The Leonard theorem (1982).

During the interval 1978–1982 Professor Eiichi Bannai (then at Ohio State University) taught a sequence of graduate courses on commutative association schemes $\Gamma = (X, \{R_i\}_{i=0}^{D})$.

The courses focussed on the P-**polynomial property** (DRG) and the Q-**polynomial property**.

As previously shown by Delsarte, if Γ is both P-polynomial and Q-polynomial, then there exist two finite sequences of orthogonal polynomials $\{u_i(x)\}_{i=0}^{D}$ and $\{u_i^*(x)\}_{i=0}^{D}$ that are related by what is called **Askey-Wilson duality** or **Delsarte duality**.

The Leonard theorem, cont.

In 1979, **Richard Askey** and his student **James Wilson** discovered some orthogonal polynomials called the q-**Racah polynomials**, that satisfy Askey-Wilson duality.

In one of Bannai's classes, there was a graduate student named **Douglas Leonard** (now at Auburn U.)

With the encouragement of Askey and Bannai, Leonard showed that the q-**Racah polynomials** give the **most general** orthogonal polynomial system that satisfies Askey-Wilson duality, under the assumption that $D \geq 9$.

The Leonard theorem, cont.

At that time, Bannai and Ito were writing a book.

In that book they included a version of Leonard's theorem that removes the assumption on D and explicitly describes all the limiting cases that show up.

The famous list of solutions consists of the following polynomials:

q-**Racah,** q-**Hahn, dual** q-**Hahn,** q-**Krawtchouk, dual** q-**Krawtchouk, quantum** q-**Krawtchouk, affine** q-**Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito.**

6. The book Algebraic Combinatorics I: Association Schemes, by **E. Bannai and T. Ito** (1985).

In 1985, Eiichi Bannai and Tatsuro Ito published the above book on Association Schemes.

The book had a **profound influence** on me.

It was from this book that I learned about the **Krein condition**, the Q-**polynomial property**, the **Norton algebra**, the **Leonard theorem**, and many other topics.

It is fair to say that **this single book set the direction for my career**.

Bannai and Ito, I thank you beyond words for writing this book, and more generally for your support throughout my career.

7. Spin Models (1989).

The concept of a **spin model** was introduced in 1989 by **V. F. R. Jones**, in his study of **knot invariants**.

A spin model is a symmetric matrix with complex entries, that satisfies two conditions called **type II** and **type III**.

The type II condition asserts that every entry is nonzero, and the **entrywise inverse** of the matrix is a **scalar multiple of the usual inverse** of the matrix.

The type III condition can be expressed using a **braid relation**.

Spin models become relevant to **algebraic graph theory** as follows.

In his original paper, Jones described three examples of spin models: the **Potts model**, the **square model**, and the **odd cyclic model**.

In 1992 **François Jaeger** showed that these spin models are in the **Bose-Mesner algebra** of a distance-regular graph.

Since that discovery, other authors found spin models in the Bose-Mesner algebra of the **even cycle (Bannai and Bannai)** the **Hadamard graphs (K. Nomura)** and the **double cover of the Higman-Sims graph (A. Munemasa)**.

Spin models, cont.

From 1995 up to the present day, the distance-regular graphs that support a spin model have been thoroughly studied by **J. Caughman, B. Curtin, F. Jaeger, K. Kawagoe, A. Munemasa, K. Nomura, Y. Watatani, N. Wolff.**

Remarkably, the spin model concept lead to numerous advances **beyond graph theory** and at a **purely algebraic level**.

For example, the spin model concept motivated the notions of a **spin Leonard pair (Curtin 2007)**, a **modular Leonard triple (Curtin 2007)**, and the **pseudo intertwiners of a Leonard triple of q-Racah type (Ter 2017)**.

8. BCN (1989).

In 1989 the authors **Andries Brouwer, Arjeh Cohen, and Arnold Neumaier** published the book **Distance-Regular Graphs**.

The book is more **encyclopedia** than **textbook**.

The book is **extremely useful**. If I need to cite a fact about distance-regular graphs, then **for sure it can be found in BCN**.

BCN, cont.

BCN did **much more** than just organize and present the existing theory.

BCN **improved the existing theory** in countless ways.

For example, the concept of **classical parameters** is introduced in BCN.

In my own research, **I use the concept of classical parameters every day**.

9. The Askey-Wilson algebra (1991).

In 1991 **Alexei Zhedanov** published the paper

Hidden symmetry of Askey-Wilson polynomials.

In that paper Zhedanov introduced the **Askey-Wilson algebra** (often called the **Zhedanov algebra**).

The Askey-Wilson algebra, cont.

For the AW algebra **the precise definition has evolved over time**, as everyone improved their understanding.

For example:

- in 1995 **P. Wiegmann** and **A. Zabrodin** gave a $\mathbb{Z}_3$-**symmetric** presentation;
- In 1999, **D. Bullock** and **J. Przytycki** gave a $\mathbb{Z}_3$-symmetric presentation that **displayed extra detail about the coefficients**;
- In 2011, Ter gave a **universal version** in which the coefficients are viewed as central elements in the algebra.

Here is the universal version.

Definition

The **(universal) AW algebra** is defined by generators and relations in the following way. The generators are A, B, C. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in the algebra.

The universal AW algebra is related to the q-Onsager algebra O_q as follows:

there exists an algebra homomorphism $\natural$ from O_q to the universal AW algebra that sends

$$A \mapsto A, \qquad B \mapsto B.$$

One **striking feature** of the universal AW algebra is that it supports an action of the **modular group** $\mathrm{PSL}_2(\mathbb{Z})$ as a group of automorphisms.

Recall that $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation with two generators ρ, σ and two relations

$$\rho^3 = 1, \qquad \sigma^2 = 1.$$

The Askey-Wilson algebra, cont.

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the universal AW algebra as a group of automorphisms in the following way:

u	A	B	C
$\rho(u)$	B	C	A
$\sigma(u)$	B	A	$C + \frac{AB-BA}{q-q^{-1}}$

Shortly, we will return to this action.

Over the decades, the AW algebra has found applications to many topics, such as

- the subconstituent algebra,
- Leonard pairs,
- orthogonal polynomials,
- skein algebras,
- the DAHA of type $(C_1^\vee, C_1)$.

These applications are summarized in the 2021 paper

Crampé, Frappat, Gaboriaud, d'Andecy, Ragoucy, and Vinet.

The Askey-Wilson algebra and its avatars.

10. The Nomura algebra (1997).

We mentioned earlier the search for spin models inside the Bose-Mesner algebra of an association scheme.

In 1997 **Kazumasa Nomura** used **any spin model** W to **canonically construct an association scheme** whose Bose-Mesner algebra contains W.

The resulting Bose-Mesner algebra is denoted $N(W)$ and called the **Nomura algebra** of W.

Nomura's construction shows how the theory of spin models is **very closely linked** to the theory of association schemes.

It is a deep open problem to **classify the association schemes that arise from Nomura's construction**.

11. K. Tanabe's study of the Doob graphs (1997).

In 1997 **Kenichiro Tanabe** published the paper:

The irreducible modules of the Terwilliger algebras of Doob schemes.

This paper had a big impact for the following reasons.

Tanabe's study of the Doob graphs, cont.

Recall that a Doob graph is a **Cartesian product** of abitrarily many copies of the **Shrikhande graph** and abitrarily many copies of the **complete graph** K_4.

In his paper Tanabe described the irreducible T-modules of a Doob graph, and found that they are **not thin** in general.

Instead, the irreducible T-modules had a **tensor product** structure that resembled certain irreducible modules of the **affine Lie algebra** $\widehat{\mathfrak{sl}_2}$.

Tanabe's study of the Doob graphs, cont.

From that hint, it was not difficult to guess that in general the T-algebra (and also the q-Onsager algebra) should be related to the **quantum affine algebra** $U_q(\widehat{\mathfrak{sl}_2})$.

At that time, I knew **almost nothing** about quantum groups, but this gave me a **big motivation to learn**.

My first teacher was **Tanabe himself**, who had earlier attended a course by **Michio Jimbo**.

Now, after many years and by the work of many people including

Tatsuro Ito, Pascal Baseilhac, Stefan Kolb

we can say that the q-**Onsager algebra** O_q **is isomorphic to a coideal subalgebra of** $U_q(\widehat{\mathfrak{sl}_2})$.

12. $(n+1, m+1)$-**hypergeometric functions** (2004).

In 2004, **Hiroshi Mizukawa** and **Hajime Tanaka** published the following influential paper:

$(n+1, m+1)$-**hypergeometric functions associated to character algebras**.
Proc. Amer. Math. Soc.

In their paper, Mizukawa and Tanaka used **character algebras** to obtain generalized hypergeometric functions that correspond to certain **multivariable Krawtchouk polynomials**.

To explain the significance (using an approach of **Plamen Iliev**), consider the **Lie algebra** $L = \mathfrak{sl}_{n+1}(\mathbb{C})$.

Let V denote a finite-dimensional irreducible L-module such that **every weight space has dimension one**.

Generalized hypergeometric functions, cont.

Let H, H^* denote **Cartan subalgebras of L that together generate L**.

Consider the **transition matrices** between an H-**weight basis** for V and an H^*-**weight basis** for V.

The entries of these transition matrices are given by the above generalized hypergeometric functions.

Generalized hypergeometric functions, cont.

For the special case $L = \mathfrak{sl}_2(\mathbb{C})$, the above matrix entries become the familiar ${}_2F_1$-**hypergeometric functions** attached to the familiar **Krawtchouk polynomials**.

13. The twisted Grassmann graphs (2005).

In 2005 **Edwin Van Dam** and **Jack Koolen** published the following influential paper:

A new family of distance-regular graphs with unbounded diameter.
Invent. Math.

The twisted Grassmann graphs, cont.

Fix integers $N \geq 2D \geq 4$.

Recall that the **Grassmann graph** $J_q(N, D)$ has vertex set X consisting of the D-dimensional subspaces of $\mathbb{V}$, where $\mathbb{V}$ is an N-dimensional vector space over the finite field $\mathrm{GF}(q)$.

Two vertices in $J_q(N, D)$ are adjacent whenever their intersection has dimension $D - 1$.

The graph $J_q(N, D)$ is distance-regular with diameter D.

The twisted Grassmann graphs, cont.

In their paper, Van Dam and Koolen construct a distance-regular graph called the **twisted Grassmann graph**, that has the same intersection numbers as $J_q(2D+1, D)$ but is not isomorphic to $J_q(2D+1, D)$.

This **very clever construction** gives the first known family of **non-vertex-transitive distance-regular graphs with unbounded diameter**.

The twisted Grassmann graphs, cont.

It is a deep open problem to **classify the Q-polynomial distance-regular graphs**.

The construction reveals the **difficulty that we face** as we investigate this problem; **esoteric new examples keep showing up!**

14. The Drinfeld polynomial of a tridiagonal pair (2010).

Concerning the classification of tridiagonal pairs, a **key idea** was supplied by **Tatsuro Ito** when he introduced the **Drinfeld Polynomial** concept around 2010.

Roughly speaking, each tridiagonal pair A, A^* of diameter D is attached to a certain polynomial of degree D, called the **Drinfeld polynomial of** A, A^*.

The Drinfeld polynomial of a tridiagonal pair, cont.

The isomorphism class of A, A^* is determined by the **eigenvalues of** A, the **eigenvalues of** A^*, and the **roots of the Drinfeld polynomial**.

To complete the classification of tridiagonal pairs, we **used the Drinfeld polynomial to confirm that no tridiagonal pair was missed by our constructions**.

The Drinfeld polynomial of a tridiagonal pair, cont.

It is fair to say that **without the Drinfeld polynomial idea, we never would have completed the classification**.

Tatsuro Ito, I thank you beyond words for your contribution!

15. The Lusztig automorphisms of the q-Onsager algebra (2020).

In 2020, **Pascal Baseilhac** and **Stefan Kolb** published the following influential paper:

Braid group action and root vectors for the q-Onsager algebra.

Transform. Groups.

Recall that the q-Onsager algebra O_q has generators A, B that satisfy the q-Dolan/Grady relations.

By construction, O_q has an automorphism that swaps $A \leftrightarrow B$.

In their paper, Baseilhac and Kolb introduce two **nonobvious automorphisms** L**,** L^* of O_q, that do the following.

We have

$$L(A) = A, \qquad L(B) = B + \frac{[A,[A,B]_q]}{(q-q^{-1})(q^2-q^{-2})},$$

$$L^*(B) = B, \qquad L^*(A) = A + \frac{[B,[B,A]_q]}{(q-q^{-1})(q^2-q^{-2})}.$$

The Lusztig automorphisms of O_q, cont.

We call L, L^* the **Lusztig automorphisms** of O_q.

In their paper, Baseilhac and Kolb use L, L^* to define **root vectors** which give rise to a **PBW basis** for O_q.

This PBW basis has been cited many times, by myself and also

Owen Goff, Ming Lu, Weiqiang Wang, Weinan Zhang.

The Lusztig automorphisms of O_q, cont.

Let me finish by explaining how L, L^* are related to the $\mathrm{PSL}_2(\mathbb{Z})$-action on the universal Askey-Wilson algebra UAW.

Recall the generators ρ, σ of $\mathrm{PSL}_2(\mathbb{Z})$.

Recall the algebra homomorphism $\natural : O_q \to UAW$.

The Lusztig automorphisms of O_q, cont.

The following diagrams commute:

$$
\begin{array}{ccc}
O_q & \xrightarrow{\natural} & UAW \\
{\scriptstyle L}\downarrow & & \downarrow{\scriptstyle (\rho^2\sigma)^2} \\
O_q & \xrightarrow[\natural]{} & UAW
\end{array}
\qquad\qquad
\begin{array}{ccc}
O_q & \xrightarrow{\natural} & UAW \\
{\scriptstyle L^*}\downarrow & & \downarrow{\scriptstyle (\sigma\rho^2)^2} \\
O_q & \xrightarrow[\natural]{} & UAW
\end{array}
$$

The Lusztig automorphisms of O_q, cont.

It is a deep open problem to find the **combinatorial signifance of the Lusztig automorpisms** at the level of Q-**polynomial distance-regular graphs**.

Conclusion

In this talk, I discussed **15 major events and conceptual breakthroughs** that had a large impact on my own work.

Along the way, we saw some **deep open problems** that should motivate **many future breakthroughs!**

It will be exciting to see what our community will discover next!

THANK YOU FOR YOUR ATTENTION!