Tridiagonal pairs, alternating elements, and distance-regular graphs

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Overview

We first describe the **alternating elements** of the algebra U_q^+.

We then review a linear algebraic object called a **tridiagonal pair**.

Roughly speaking, this is a pair A, A^* of diagonalizable linear maps on a nonzero finite-dimensional vector space V, that each act on the eigenspaces of the other one in a (block) tridiagonal fashion.

We will use a **tetrahedron diagram** to describe six direct sum decompositions of V...

Overview, cont.

We will impose a condition on A, A^* under which V becomes an irreducible U_q^+-module.

In our main results, we describe how the alternating elements of U_q^+ act on the above six decompositions of V.

Finally, we improve our results under the assumption that the pair A, A^* comes from a certain type of **distance-regular graph**.

A remarkable fact

We start by describing a remarkable fact.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $\mathbb{F}$ denote a field.

Fix a nonzero scalar $b \in \mathbb{F}$ that is not a root of unity.

A remarkable fact, cont.

Define an algebra over $\mathbb{F}$ by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_b]_{b^{-1}}] = 0,$$
$$[W_1, [W_1, [W_1, W_0]_b]_{b^{-1}}] = 0,$$

where

$$[X, Y] = XY - YX, \qquad [X, Y]_b = bXY - YX.$$

A remarkable fact, cont.

Using W_0, W_1 and the equations to come, we recursively define some elements

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

in the following order:

$$\begin{array}{llll} W_0, & W_1, & G_1, & \tilde{G}_1, \\ W_{-1}, & W_2, & G_2, & \tilde{G}_2, \\ W_{-2}, & W_3, & G_3, & \tilde{G}_3, \quad \ldots \end{array}$$

A remarkable fact, cont.

For $n \geq 1$,

$$G_n = \frac{\sum_{k=0}^{n-1} W_{-k} W_{n-k} b^{-k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} b^{-k}}{1 + b^{-n}} + \frac{[W_n, W_0]}{(1 + b^{-n})(1 - b^{-1})},$$

$$\tilde{G}_n = G_n + \frac{[W_0, W_n]}{1 - b^{-1}},$$

$$W_{-n} = \frac{[W_0, G_n]_b}{b - 1}, \qquad W_{n+1} = \frac{[G_n, W_1]_b}{b - 1}.$$

A remarkable fact, cont.

The remarkable fact [Ter 2019] is that for $k, \ell \in \mathbb{N}$,

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0,$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

A remarkable fact, cont.

The defining relations that we started with, are called the q-**Serre relations**, where $q^2 = b$.

The defined algebra is denoted by U_q^+, and called the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

The elements

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

are called the **alternating elements** of U_q^+.

The origin of the alternating elements

Next we explain how the alternating elements get their name.

We will refer to the q-**shuffle algebra** realization of U_q^+, which was introduced by **Marc Rosso** in 1995.

For the q-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described next.

The free algebra $\mathbb{U}$

Let x, y denote noncommuting indeterminates.

Let $\mathbb{U}$ denote the free algebra over $\mathbb{F}$ that has generators x, y.

By a **letter** in $\mathbb{U}$ we mean x or y.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{U}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{U}$ has a basis consisting of its words.

The q-shuffle product on $\mathbb{U}$

We just defined the free algebra $\mathbb{U}$.

Next we endow $\mathbb{U}$ with a q-**shuffle** product, denoted $\star$.

This q-shuffle product is due to M. Rosso.

The q-shuffle product on $\mathbb{U}$, cont.

For letters u, v we have

$$u \star v = uv + vuq^{\langle u,v \rangle}$$

where

$\langle\, , \,\rangle$	x	y
x	2	−2
y	−2	2

So

$$x \star y = xy + q^{-2} yx, \qquad y \star x = yx + q^{-2} xy,$$

$$x \star x = (1 + q^2) xx, \qquad y \star y = (1 + q^2) yy.$$

The q-shuffle product on $\mathbb{U}$, cont.

For words u, v in $\mathbb{U}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$\begin{aligned} u \star v = \; & a_1 a_2 b_1 b_2 \\ & + a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\ & + a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle} \\ & + b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ & + b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \end{aligned}$$

Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathbb{U}$ into an (associative) algebra.

The algebra U

Definition

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{U}$ generated by x, y.

The algebra U is described as follows. We have

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0,$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1}) = b + b^{-1} + 1$.

So in the q-shuffle algebra $\mathbb{U}$ the elements x, y satisfy the q-Serre relations.

How U_q^+ is related to U.

Consequently, there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{U}$, that sends $W_0 \mapsto x$ and $W_1 \mapsto y$.

The map $\natural$ has image U by construction.

Theorem (Rosso, 1995)

The map $\natural : U_q^+ \to U$ is an algebra isomorphism.

The alternating elements of U_q^+, revisited

We now apply the map $\natural : U_q^+ \to U$ to the alternating elements.

Lemma (Ter 2019)

The map $\natural$ *sends*

$$
\begin{array}{llll}
W_0 \mapsto x, & W_{-1} \mapsto xyx, & W_{-2} \mapsto xyxyx, & \ldots \\
W_1 \mapsto y, & W_2 \mapsto yxy, & W_3 \mapsto yxyxy, & \ldots \\
G_1 \mapsto yx, & G_2 \mapsto yxyx, & G_3 \mapsto yxyxyx, & \ldots \\
\tilde{G}_1 \mapsto xy, & \tilde{G}_2 \mapsto xyxy, & \tilde{G}_3 \mapsto xyxyxy, & \ldots
\end{array}
$$

More relations for the alternating elements

There are four kinds of alternating elements, and we stated that the alternating elements of each kind mutually commute.

The alternating elements satisfy many additional relations, as we now explain.

For notational convenience, define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Relations between the alternating elements, I

Lemma (Type I relations)

For $k \in \mathbb{N}$ the following relations hold in U_q^+:

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - b^{-1})(\tilde{G}_{k+1} - G_{k+1}),$$
$$[W_0, G_{k+1}]_b = [\tilde{G}_{k+1}, W_0]_b = (b - 1)W_{-k-1},$$
$$[G_{k+1}, W_1]_b = [W_1, \tilde{G}_{k+1}]_b = (b - 1)W_{k+2}.$$

Relations between the alternating elements, II

Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0,$$
$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,$$
$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0,$$
$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$
$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0,$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$
$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

Lemma (Type III relations)

For $n \geq 1$ the following relations hold in U_q^+:

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{n-1-2k},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{2k+1-n},$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{2k+1-n}.$$

We remark that the previous relations are similar to some relations for the q-Onsager algebra, that were found earlier by the physicists **P. Baseilhac, K. Koizumi, K. Shigechi** [2005], [2010] in their study of integrable models in statistical mechanics.

We discovered the alternating elements of U_q^+ by using the above works as a guide.

We have been discussing the alternating elements of U_q^+.

Next, we apply these alternating elements to the theory of **tridiagonal pairs**.

As a warmup, let us recall the definition of a tridiagonal pair.

Tridiagonal pairs, preliminaries

Let V denote a nonzero vector space over $\mathbb{F}$ with finite dimension.

Let $\mathrm{End}(V)$ denote the algebra over $\mathbb{F}$, consisting of the $\mathbb{F}$-linear maps from V to V.

Consider an ordered pair A, A^* of maps in $\mathrm{End}(V)$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** (or **TD pair**) whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

Comments

Referring to the above definition, it turns out that $d = \delta$; we call this common value the **diameter** of the TD pair.

According to a common notational convention, A^* denotes the conjugate-transpose of A.

We are not using this convention.

In a TD pair, the elements A and A^* are arbitrary subject to the conditions (i)–(iv) above.

History and connections

The TD pairs were introduced in 2001 by Ito, Tanabe, and Terwilliger.

The TD pairs over an algebraically closed field were classified up to isomorphism by Ito, Nomura, and Terwilliger (2011).

TD pairs are related to:

- Q-polynomial distance-regular graphs,
- the orthogonal polynomials of the Askey scheme,
- the Askey-Wilson, Onsager, and q-Onsager algebras,
- the double affine Hecke algebra of type $(C_1^\vee, C_1)$,
- the Lie algebras $\mathfrak{sl}_2$ and $\widehat{\mathfrak{sl}}_2$,
- the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{\mathfrak{sl}}_2)$,
- integrable models in statistical mechanics.

Tridiagonal systems

In our study of tridiagonal pairs, it is useful to employ a related object called a **tridiagonal system**.

Before defining this object, we review some concepts.

Standard orderings

Let A, A^* denote a TD pair on V. An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d).$$

If the ordering $\{V_i\}_{i=0}^d$ is standard then the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard.

Similar comments apply to A^*.

Primitive idempotents

Given an eigenspace W of a diagonalizable linear map, the corresponding **primitive idempotent** acts on W as the identity map, and acts on the other eigenspaces as zero.

The definition of a tridiagonal system

Definition

By a **tridiagonal system** (or **TD system**) on V, we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$$

such that

(i) A, A^* is a TD pair on V;

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

The D_4 action

Consider a TD system $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ on V.

Each of the following is a TD system on V:

$$\Phi^* = (A^*, \{E_i^*\}_{i=0}^d, A, \{E_i\}_{i=0}^d);$$
$$\Phi^\downarrow = (A, \{E_i\}_{i=0}^d, A^*, \{E_{d-i}^*\}_{i=0}^d);$$
$$\Phi^\Downarrow = (A, \{E_{d-i}\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d).$$

The D_4 action, cont.

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all TD systems,

$$*^2 = 1, \qquad \downarrow^2 = 1, \qquad \Downarrow^2 = 1,$$
$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \downarrow\Downarrow.$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the above relations is called the **dihedral group** D_4. Recall that D_4 is the group of symmetries of a square, and has 8 elements.

The elements $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all TD systems.

TD systems in the same D_4-orbit are called **relatives**.

Until further notice, fix a TD system on V:

$$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d).$$

The eigenvalues

Definition

For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A corresponding to E_i. For $0 \leq i \leq d$ let θ^*_i denote the eigenvalue of A^* corresponding to E^*_i.

Definition

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

The Φ-split decomposition

By a **decomposition of** V we mean a sequence $\{\mathcal{V}_i\}_{i=0}^d$ of nonzero subspaces whose direct sum is V.

For example, the sequences $\{E_i V\}_{i=0}^d$ and $\{E_i^* V\}_{i=0}^d$ are decompositions of V.

Next we consider another decomposition of V, called the **Φ-split decomposition**.

This decomposition is described on the next slides.

The Φ-split decomposition, cont.

Definition

For $0 \leq i \leq d$ define

$$U_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{d-i} V).$$

For example,

$$U_0 = E_0^* V, \qquad U_d = E_0 V.$$

The Φ-split decomposition, cont.

Lemma (Ito, Tanabe, Ter 2001)

The sequence $\{U_i\}_{i=0}^d$ is a decomposition of V.

We call $\{U_i\}_{i=0}^d$ the **Φ-split decomposition of** V.

The Φ-split decomposition, cont.

The maps A and A^* act on the Φ-split decomposition as follows.

Lemma (Ito, Tanabe, Ter 2001)

We have

(i) $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$ $(1 \leq i \leq d)$;

(ii) $(A^* - \theta_0^* I)U_0 = 0$;

(iii) $(A - \theta_{d-i} I)U_i \subseteq U_{i+1}$ $(0 \leq i \leq d-1)$;

(iv) $(A - \theta_0 I)U_d = 0$.

We give another version of the above result.

Corollary

We have

(i) $A^* U_i \subseteq U_{i-1} + U_i \ (1 \leq i \leq d)$;

(ii) $A^* U_0 \subseteq U_0$;

(iii) $AU_i \subseteq U_i + U_{i+1} \ (0 \leq i \leq d-1)$;

(iv) $AU_d \subseteq U_d$.

The Φ-split decomposition, cont.

The next result clarifies how the $\{U_i\}_{i=0}^d$ are related to the eigenspaces of A and A^*.

Lemma (Ito, Tanabe, Ter 2001)

For $0 \le i \le d$,

(i) $U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V$;

(ii) $U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V$.

Replacing Φ by a relative

We have been discussing the Φ-split decomposition of V.

If we replace Φ by a relative, then we get another decomposition of V.

On the next slide, we name the resulting decompositions in a uniform way.

Six decompositions of V

Example

In each row of the table below, we display a decomposition of V.

decomp. name	i^{th} component of the decomposition
$[0, D]$	$E_i V$
$[0^*, D^*]$	$E^*_i V$
$[0^*, 0]$	$(E^*_0 V + \cdots + E^*_i V) \cap (E_0 V + \cdots + E_{d-i} V)$
$[0^*, D]$	$(E^*_0 V + \cdots + E^*_i V) \cap (E_i V + \cdots + E_d V)$
$[D^*, 0]$	$(E^*_{d-i} V + \cdots + E^*_d V) \cap (E_0 V + \cdots + E_{d-i} V)$
$[D^*, D]$	$(E^*_{d-i} V + \cdots + E^*_d V) \cap (E_i V + \cdots + E_d V)$

Six decompositions of V, cont.

In the next result, we clarify how the above six decompositions are related to the eigenspaces of A and A^*.

Lemma

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V from the previous example. Then for $0 \le i \le d$ the sums $\mathcal{V}_0 + \cdots + \mathcal{V}_i$ and $\mathcal{V}_i + \cdots + \mathcal{V}_d$ are given in the table below.

decomp. name	$\mathcal{V}_0 + \cdots + \mathcal{V}_i$	$\mathcal{V}_i + \cdots + \mathcal{V}_d$
$[0, D]$	$E_0V + \cdots + E_iV$	$E_iV + \cdots + E_dV$
$[0^*, D^*]$	$E^*_0V + \cdots + E^*_iV$	$E^*_iV + \cdots + E^*_dV$
$[0^*, 0]$	$E^*_0V + \cdots + E^*_iV$	$E_0V + \cdots + E_{d-i}V$
$[0^*, D]$	$E^*_0V + \cdots + E^*_iV$	$E_iV + \cdots + E_dV$
$[D^*, 0]$	$E^*_{d-i}V + \cdots + E^*_dV$	$E_0V + \cdots + E_{d-i}V$
$[D^*, D]$	$E^*_{d-i}V + \cdots + E^*_dV$	$E_iV + \cdots + E_dV$

Six decompositions of V, cont.

Next, we describe the actions of A and A^* on the above six decompositions of V.

Lemma

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V from the previous example. Then for $0 \leq i \leq d$ the actions of A and A^* on $\mathcal{V}_i$ are described in the table below.

decomp. name	action of A on $\mathcal{V}_i$	action of A^* on $\mathcal{V}_i$
$[0, D]$	$(A - \theta_i I)\mathcal{V}_i = 0$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, D^*]$	$A\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$	$(A^* - \theta^*_i I)\mathcal{V}_i = 0$
$[0^*, 0]$	$(A - \theta_{d-i} I)\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$	$(A^* - \theta^*_i I)\mathcal{V}_i \subseteq \mathcal{V}_{i-1}$
$[0^*, D]$	$(A - \theta_i I)\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$	$(A^* - \theta^*_i I)\mathcal{V}_i \subseteq \mathcal{V}_{i-1}$
$[D^*, 0]$	$(A - \theta_{d-i} I)\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$	$(A^* - \theta^*_{d-i} I)\mathcal{V}_i \subseteq \mathcal{V}_{i-1}$
$[D^*, D]$	$(A - \theta_i I)\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$	$(A^* - \theta^*_{d-i} I)\mathcal{V}_i \subseteq \mathcal{V}_{i-1}$

Six decompositions of V, cont.

Here is another version of the above result.

Corollary

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V from the previous example. Then for $0 \le i \le d$ the actions of A and A^* on $\mathcal{V}_i$ are described in the table below.

decomp. name	action of A on $\mathcal{V}_i$	action of A^* on $\mathcal{V}_i$
$[0, D]$	$A\mathcal{V}_i \subseteq \mathcal{V}_i$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, D^*]$	$A\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_i$
$[0^*, 0]$	$A\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[0^*, D]$	$A\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[D^*, 0]$	$A\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[D^*, D]$	$A\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$A^*\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$

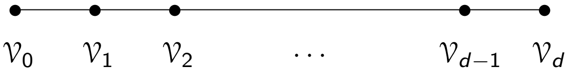
The tetrahedron diagram

We have been discussing six decompositions of V.

We now draw a diagram that illustrates our discussion so far.

The tetrahedron diagram

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V. We describe this decomposition by the diagram



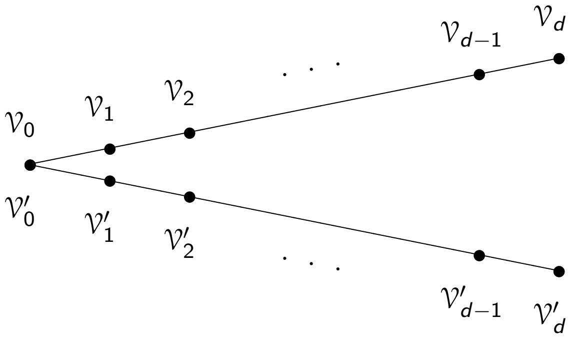
The labels $\mathcal{V}_i$ might be suppressed, if they are clear from the context.

The tetrahedron diagram, cont.

Let $\{\mathcal{V}_i\}_{i=0}^d$ and $\{\mathcal{V}'_i\}_{i=0}^d$ denote decompositions of V. The condition

$$\mathcal{V}_0 + \mathcal{V}_1 + \cdots + \mathcal{V}_i = \mathcal{V}'_0 + \mathcal{V}'_1 + \cdots + \mathcal{V}'_i \qquad (0 \leq i \leq d)$$

will be described by the diagram



The tetrahedron diagram, cont.

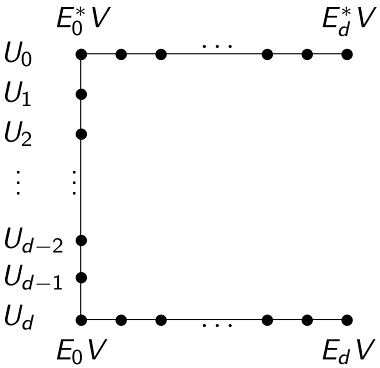
To illustrate the above diagram convention, consider the Φ-split decomposition $\{U_i\}_{i=0}^{d}$ of V.

Recall that for $0 \leq i \leq d$ we have

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V,$$
$$U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V.$$

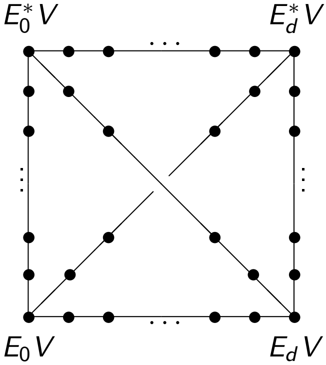
The corresponding diagram is shown on next slide:

The tetrahedron diagram, cont.



The tetrahedron diagram, cont.

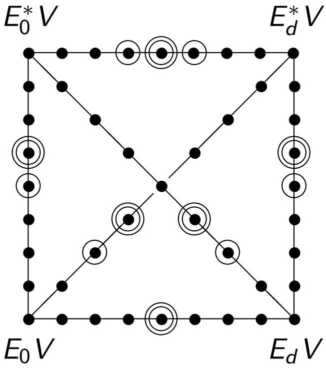
Earlier we displayed six decompositions of V. The corresponding diagram is shown below:



This diagram is called the **tetrahedron diagram** of Φ.

The following picture shows how A acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

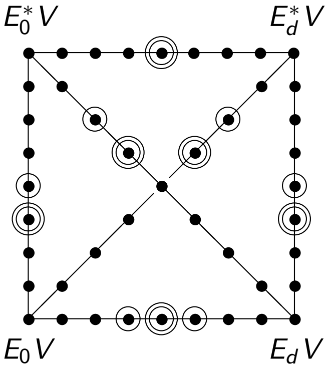
A action:



The tetrahedron diagram, cont.

The following picture shows how A^* acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

A^* action:



We will return to the tetrahedron diagram shortly.

Next we discuss the **tridiagonal relations**.

The tridiagonal relations

Theorem (Ito, Tanabe, Ter 2001)

There exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathbb{F}$ such that both

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],$$
$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A].$$

The sequence is unique if $d \geq 3$.

The above relations are called the **tridiagonal relations**.

Next we describe how the above parameters $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ are related to the eigenvalues $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$.

The tridiagonal relations, cont.

Lemma

(i) *the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are both equal to $\beta + 1$ *for* $2 \leq i \leq d-1$*;*

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ $(1 \leq i \leq d-1)$;

(iii) $\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ $(1 \leq i \leq d-1)$;

(iv) $\varrho = \theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i)$ $(1 \leq i \leq d)$;

(v) $\varrho^* = \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i)$ $(1 \leq i \leq d)$.

TD systems of q-Serre type

We now impose a condition on the TD system Φ.

Definition

The TD system Φ is said to have q-**Serre type** whenever $\theta_i = q^2\theta_{i-1}$ and $\theta^*_i = q^{-2}\theta^*_{i-1}$ for $1 \leq i \leq d$.

From now on, we assume that Φ has q-Serre type.

Abbreviate $b = q^2$.

TD systems of q-Serre type

Lemma

We have

$$\beta = b + b^{-1}, \quad \gamma = 0, \quad \gamma^* = 0, \quad \varrho = 0, \quad \varrho^* = 0.$$

Moreover the TD relations become the q-Serre relations

$$[A, [A, [A, A^*]_b]_{b^{-1}}] = 0, \qquad [A^*, [A^*, [A^*, A]_b]_{b^{-1}}] = 0.$$

Corollary

The vector space V becomes a U_q^+-module on which $W_0 = A$ and $W_1 = A^*$. The U_q^+-module V is irreducible.

Going forward, we view V as a U_q^+-module.

Our next goal, is to describe how the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram.

How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

Theorem (Ter 2022)

We refer to the U_q^+-module V. Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V from the tetrahedron diagram. Then for $k \in \mathbb{N}$ and $0 \leq i \leq d$ the actions of W_{-k} and W_{k+1} on $\mathcal{V}_i$ are described in the table below.

decomp. name	action of W_{-k} on $\mathcal{V}_i$	action of W_{k+1} on $\mathcal{V}_i$
$[0, D]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_i$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, D^*]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i$
$[0^*, 0]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[0^*, D]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[D^*, 0]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[D^*, D]$	$W_{-k}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$W_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$

How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

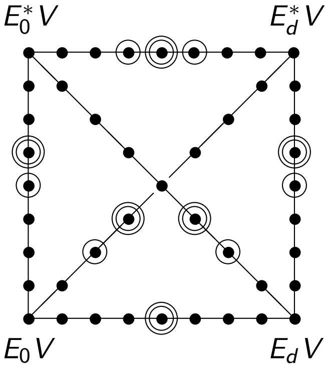
Next we use the tetrahedron diagram to illustrate the previous theorem.

Pick $k \in \mathbb{N}$.

How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

The following picture shows how W_{-k} acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

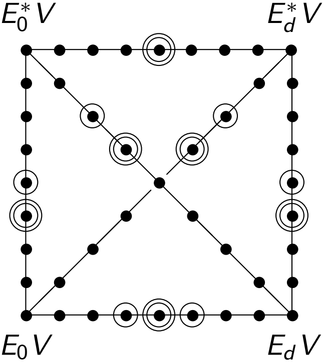
W_{-k} action:



How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

The following picture shows how W_{k+1} acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

W_{k+1} action:



How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

Theorem (Ter 2022)

We refer to the U_q^+-module V. Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V from the tetrahedron diagram. Then for $k \in \mathbb{N}$ and $0 \le i \le d$ the actions of G_{k+1} and $\tilde{G}_{k+1}$ on $\mathcal{V}_i$ are described in the table below.

decomp. name	action of G_{k+1} on $\mathcal{V}_i$	action of $\tilde{G}_{k+1}$ on $\mathcal{V}_i$
$[0, D]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, D^*]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, 0]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$
$[0^*, D]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$
$[D^*, 0]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i + \mathcal{V}_{i+1}$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i$
$[D^*, D]$	$G_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_{i-1} + \mathcal{V}_i + \mathcal{V}_{i+1}$	$\tilde{G}_{k+1}\mathcal{V}_i \subseteq \mathcal{V}_i$

How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

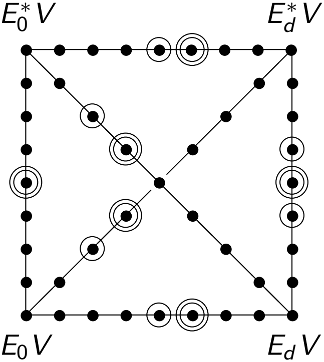
Next we use the tetrahedron diagram to illustrate the above theorem.

Pick $k \in \mathbb{N}$.

How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

The following picture shows how G_{k+1} acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

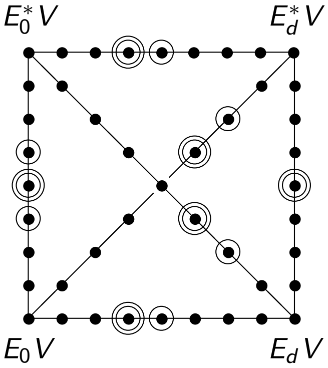
G_{k+1} action:



How the alternating elements of U_q^+ act on the six decompositions of V from the tetrahedron diagram

The following picture shows how $\tilde{G}_{k+1}$ acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

$\tilde{G}_{k+1}$ action:



It would be nice to have explicit matrix representations, for the action of the alternating elements on the U_q^+-module V.

To do this, we seek an attractive basis for V.

Attractive bases for V

Motivated by the original remarkable fact, we seek:

(i) a basis of common eigenvectors for $\{W_{-k}\}_{k\in\mathbb{N}}$;
(ii) a basis of common eigenvectors for $\{W_{k+1}\}_{k\in\mathbb{N}}$;
(iii) a basis of common eigenvectors for $\{G_{k+1}\}_{k\in\mathbb{N}}$;
(iv) a basis of common eigenvectors for $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$.

Unfortunately, the above bases might not exist.

The difficulty: the alternating elements might not be diagonalizable on V.

The good news: In certain situations, the alternating elements are diagonalizable on V.

Such a situation comes up in **algebraic graph theory**.

We now describe this situation briefly.

Distance-regular graphs

Recall the field $\mathbb{R}$ of real numbers.

From now on, assume that $\mathbb{F} = \mathbb{R}$.

In the topic of algebraic graph theory, there is a family of finite undirected graphs, said to be **distance-regular**.

Distance-regular graphs with classical parameters

There is a kind of distance-regular graph, said to have **classical parameters** (d, b, α, σ).

The parameter d is the diameter of the graph.

The parameters b, α, σ are real numbers used to describe the intersection numbers of the graph.

Distance-regular graphs with classical parameters

From now on, we fix a distance-regular graph Γ that has diameter $d \geq 3$ and classical parameters (d, b, α, σ) with $b \neq 1$ and $\alpha = b - 1$.

The condition on α implies that Γ is **formally self-dual**.

It is known that b is an integer and $b \neq 0$, $b \neq -1$.

Note that b is not a root of unity.

Some notation

Let X denote the vertex set of Γ.

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of matrices that have rows and columns indexed by X and all entries in $\mathbb{R}$.

Let $\mathbb{V} = \mathbb{R}^X$ denote the vector space consisting of the column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{R}$.

Note that $\mathrm{Mat}_X(\mathbb{R})$ acts on $\mathbb{V}$ by left multiplication.

The adjacency matrix and dual adjacency matrix

Let $\mathbb{A} \in \mathrm{Mat}_X(\mathbb{R})$ denote the adjacency matrix of Γ.

The matrix $\mathbb{A}$ is symmetric, and each entry is 0 or 1.

From now on, fix $x \in X$ and let $\mathbb{A}^* = \mathbb{A}^*(x) \in \mathrm{Mat}_X(\mathbb{R})$ denote the dual adjacency matrix of Γ with respect to x.

The matrix $\mathbb{A}^*$ is diagonal.

The subconstituent algebra $\mathbb{T}$

Let $\mathbb{T} = \mathbb{T}(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by $\mathbb{A}, \mathbb{A}^*$.

The algebra $\mathbb{T}$ is called the **subconstituent algebra** (or **Terwilliger algebra**) of Γ with respect to x.

By construction, $\mathbb{T}$ is closed under the transpose map.

The irreducible $\mathbb{T}$-modules

We comment on the $\mathbb{T}$-modules.

By a $\mathbb{T}$-**module**, we mean a subspace $V \subseteq \mathbb{V}$ such that $\mathbb{T}V \subseteq V$.

It is known that every $\mathbb{T}$-module is a direct sum of irreducible $\mathbb{T}$-modules.

In particular, the $\mathbb{T}$-module $\mathbb{V}$ is a direct sum of irreducible $\mathbb{T}$-modules.

It is known that $\mathbb{A}$, $\mathbb{A}^*$ act on each irreducible $\mathbb{T}$-module as a TD pair.

Adjusting $\mathbb{A}$ and $\mathbb{A}^*$

For convenience, we now adjust $\mathbb{A}$ and $\mathbb{A}^*$.

It is known that for $\mathbb{A}$ and $\mathbb{A}^*$ the roots of the minimal polynomial have the form

$$rb^{-i} + s \qquad\qquad (0 \le i \le d),$$

where $r, s \in \mathbb{R}$ and $r \neq 0$.

Define $A, A^* \in \mathrm{Mat}_X(\mathbb{R})$ such that

$$\mathbb{A} = A + sI, \qquad\qquad \mathbb{A}^* = A^* + sI.$$

By construction, for A and A^* the roots of the minimal polynomial are $\{rb^{-i}\}_{i=0}^{d}$.

By construction, A and A^* are symmetric.

By construction, the algebra $\mathbb{T}$ is generated by A, A^*.

It is known that

$$[A, [A, [A, A^*]_b]_{b^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_b]_{b^{-1}}] = 0.$$

These are the q-Serre relations, where q is a complex number such that $q^2 = b$.

The alternating elements in $\mathbb{T}$

Lemma

With the above notation, there exists an algebra homomorphism $U_q^+ \to \mathbb{T}$ that sends $W_0 \mapsto A$ and $W_1 \mapsto A^*$. This map is surjective.

Definition

By an **alternating element** in $\mathbb{T}$, we mean the image of an alternating element in U_q^+ under the above homomorphism.

The irreducible $\mathbb{T}$-modules

Lemma (Ter 2022)

Referring to the alternating elements in $\mathbb{T}$, the following hold for $k \in \mathbb{N}$:

(i) W_{-k} *and* W_{k+1} *are symmetric;*

(ii) G_{k+1} *and* $\tilde{G}_{k+1}$ *are the transposes of each other.*

Lemma (Ter 2022)

The alternating elements in $\mathbb{T}$ are diagonalizable on each irreducible $\mathbb{T}$-module.

We now state our final results.

Theorem (Ter 2022)

Each irreducible $\mathbb{T}$-module is a direct sum of its common eigenspaces for $\{W_{-k}\}_{k\in\mathbb{N}}$, and a direct sum of its common eigenspaces for $\{W_{k+1}\}_{k\in\mathbb{N}}$.

Theorem (Ter 2022)

Each irreducible $\mathbb{T}$-module is a direct sum of its common eigenspaces for $\{G_{k+1}\}_{k\in\mathbb{N}}$, and a direct sum of its common eigenspaces for $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$.

Summary

In this paper, we first described the alternating elements of U_q^+.

We then defined a TD pair A, A^* on V. We used a tetrahedron diagram to describe six decompositions of V.

We then assumed that A, A^* has q-Serre type, and showed how V becomes an irreducible U_q^+-module.

We then described how the alternating elements of U_q^+ act on the six decompositions from the tetrahedron diagram.

Finally, we improved our results under the assumption that the TD pair A, A^* comes from a certain type of distance-regular graph.

Thank you for your attention!

THE END