Tridiagonal pairs and applications

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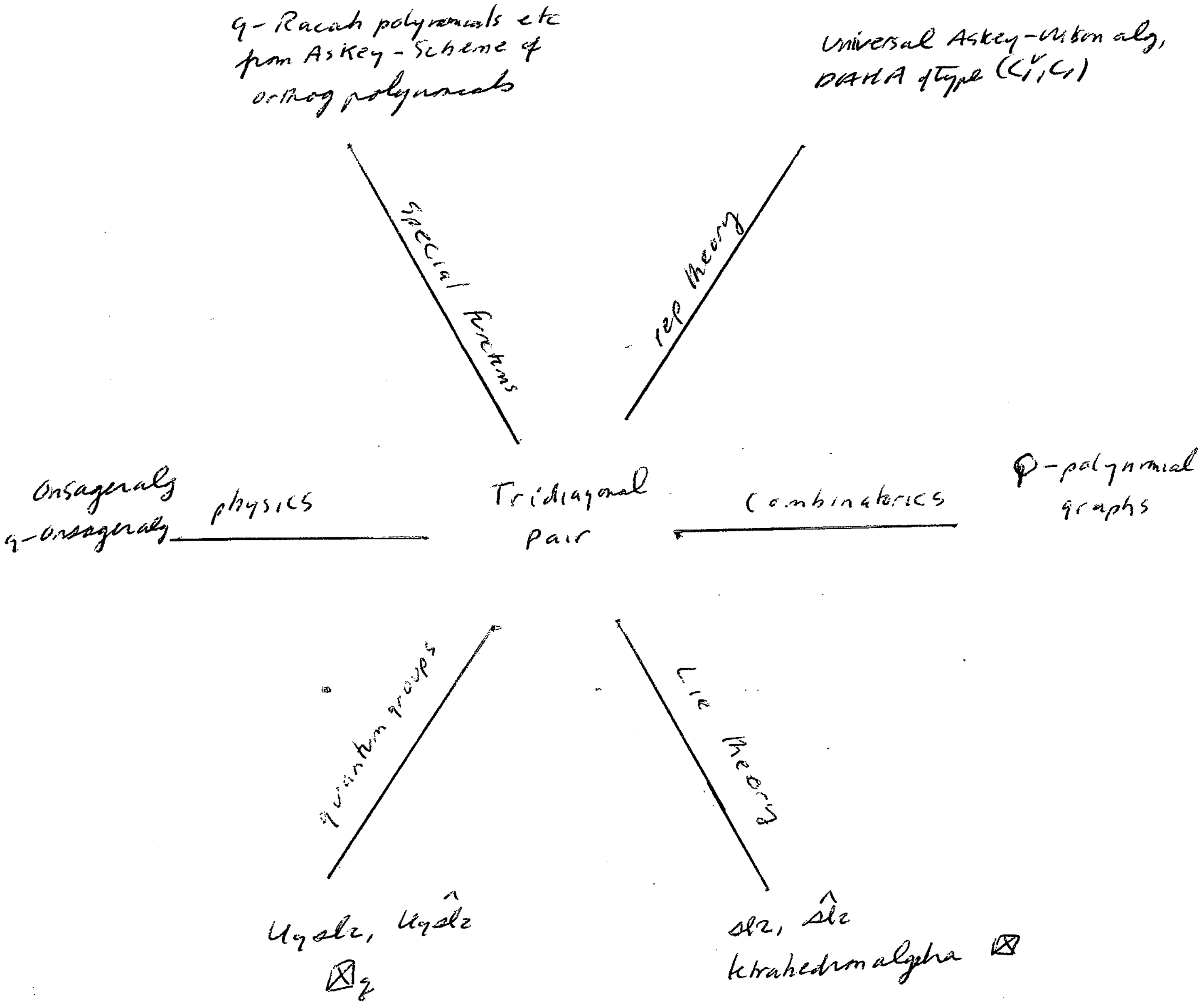
__Abstract__ A tridiagonal pair is a linear algebraic object with connections to

- combinatorics (Q-polynomial graphs)
- Lie theory (sl_2, affine sl_2, tetrahedron algebra)
- quantum groups ($U_q sl_2$, affine $U_q sl_2$, q-tetrahedron alg)
- physics (Onsager algebra, q-Onsager algebra)
- special functions (q-Racah polynomials etc from Askey scheme of orthogonal polynomials)
- representation theory (the universal Askey-Wilson algebra, the DAHA of rank 1)

We will discuss some of these connections.


q-Racah polynomials etc
from Askey-Scheme of
orthog polynomials
Universal Askey-Wilson alg,
DAHA of type $(C_1^\vee, C_1)$
special functions
rep theory
Onsager alg
q-Onsager alg
physics
Tridiagonal
pair
combinatorics
Q-polynomial
graphs
quantum groups
Lie theory
$U_q sl_2$, $U_q \hat{sl}_2$
$\boxtimes_q$
sl_2, $\hat{sl}_2$
tetrahedron algebra $\boxtimes$

I Combinatorics

Let Γ denote a finite undirected graph with vertex set X and path-length distance function ∂

Adjacency matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ has (x,y)-entry

$$A_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = 1 \\ 0 & \text{if } \partial(x,y) \neq 1 \end{cases} \qquad x, y \in X$$

Vector space $V = \mathbb{C}X$ has basis X

"standard module"

A acts $V \to V$ by

$$Ax = \sum_{\substack{y \in X \\ \partial(x,y) = 1}} y$$

A is real and sym. hence diagonalizable.

Order the dist eigenvalues of A:

$$\theta_0, \theta_1, \ldots, \theta_d$$

For $0 \le i \le d$ define

$$V_i = \text{eigenspace of } A \text{ for } \theta_i$$

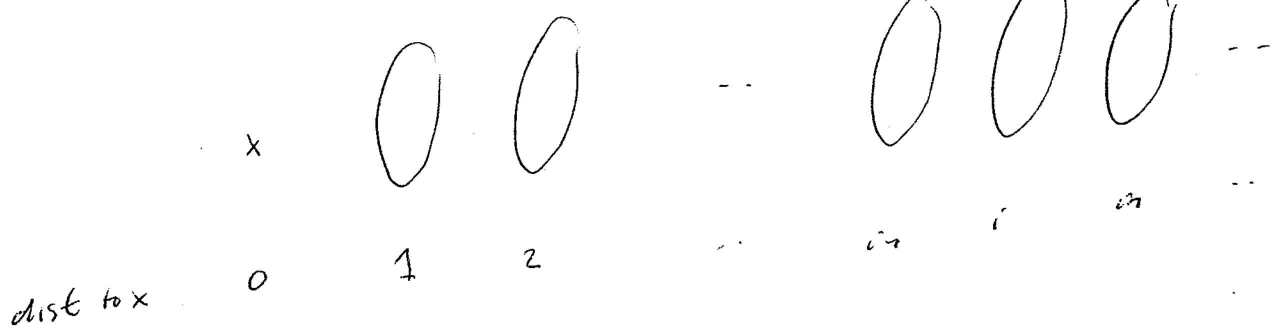
$$V = \sum_{i=0}^{d} V_i \qquad (\mathrm{ds})$$

" A-eigenspace decomp of V "

The Q-polynomial property

Fix $x \in X$

Partition X by distance to x:



For $i \geq 0$ define

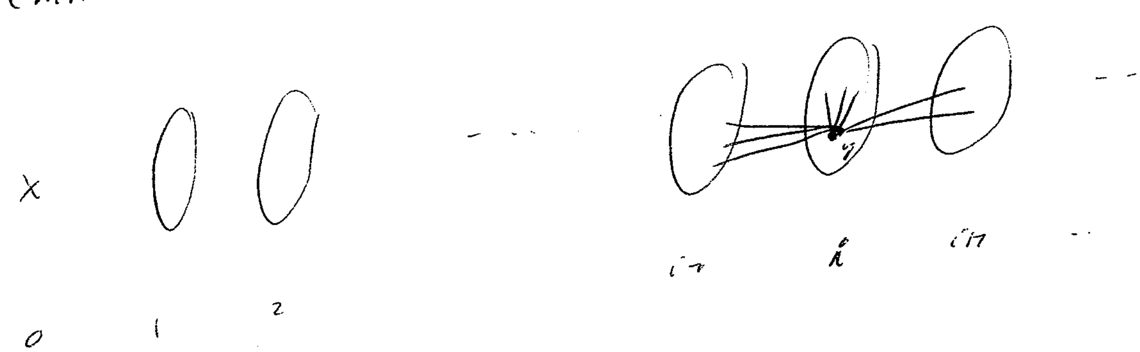
$$V_i^* = \mathrm{Span}\{y \in X \mid \partial(x,y) = i\}$$

"ith subconstituent wrt x"

So

$$V = \sum_{i \geq 0} V_i^* \qquad \text{(ds)} \qquad (\star)$$

Consider A action on $(\star)$



For $i \geq 0$,

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$$

$$(V_{-1}^* = 0)$$

DEF The vertex x is <u>Q-polynomial</u> wrt ordering $\{E_i\}_{i=0}^d$ whenever there exists a diagonal matrix $A^* = A^*(x) \in Mat_X(\mathbb{C})$ st

(i) $(\star)$ is A^*-eigenspace decomp of V,

(ii) For $0 \le i \le d$,

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$$

$$V_{-1} = 0, \quad V_{d+1} = 0$$

"dual adjacency matrix"

Call Γ <u>Q-polynomial</u> whenever Γ has a Q-polynomial vertex.

Many examples:

hyper-cubes

cycles

Johnson graphs

Grassman graphs

dual polar graphs

...

See Book

Brouwer Cohen Neumaier Distance-Regular Graphs

Research topic

What is the combinatorial meaning of Q-polynomial?

For a Q-poly vertex x, let $T = T(x)$ denote the subalg of $Mat_X(\mathbb{C})$ gen by A, A^*

"subconstituent algebra"

T is ss but not commutative

Decompose V into a dir sum of irred T-modules

What does this decomp tell us abt Γ ?

Ex Γ = hypercube of dimension d

Here $\theta_i = d - 2i$ $\quad 0 \le i \le d$

Fix $x \in X$, Define diagonal matrix $A^* \in Mat_X(\mathbb{C})$

with (y,y)-entry

$$A^*_{yy} = d - 2i \qquad i = \partial(x,y) \qquad y \in X$$

Then A^* is a dual adjacency matrix for x.

By Junie Go (2000)

$$T \cong Mat_{d+1}(\mathbb{C}) \oplus Mat_{d-1}(\mathbb{C}) \oplus Mat_{d-3}(\mathbb{C}) \oplus \cdots$$

$$\dim T = (d+1)^2 + (d-1)^2 + (d-3)^2 + \cdots$$

$$= \binom{d+3}{3}$$

II Algebra

Field $\mathbb{F}$ alg closed, char 0

$0 \neq V$ = f.d. vector space over $\mathbb{F}$

Given $A, A^* \in \mathrm{End}(V)$

Call A, A^* a <u>TD pair</u> whenever:

(i) each of A, A^* is diagonalizable

(ii) $\exists$ ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A st

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \leq i \leq d$$

$$(V_{-1} = 0, \quad V_{d+1} = 0)$$

(iii) $\exists$ ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* st

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \leq i \leq \delta$$

$$(V_{-1}^* = 0, \quad V_{\delta+1}^* = 0)$$

(iv) $\nexists$ subspace $W \subseteq V$ st

$W \neq 0, \quad W \neq V, \quad AW \subseteq W, \quad A^* W \subseteq W$

The TD pairs are classified up to iso

Ito, Nomura, Ter 2012

<u>Research topic</u> investigate connections to

Lie theory + quantum groups etc.

Assume A, A^* is a TD pair

turns out $d = \delta$ "diameter"

For $0 \le i \le d$ let

θ_i = eigenvalue of A for V_i

θ_i^* = — — A^* — V_i^*

Turns out

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and indep of i for $2 \le i \le d-1$

☆

Some solutions to ☆

θ_i	θ_i^*	Name
$d-2i$	$d-2i$	Krawtchouk
q^{d-2i}	q^{d-2i}	q-geometric
$aq^{d-2i}+a^{-1}q^{2i-d}$	$bq^{d-2i}+b^{-1}q^{2i-d}$	q-Racah

$0 \neq q$ not a root of 1

$a \neq 0,\ b \neq 0$

Ex For $d \geq 0$ define

$$A = \begin{pmatrix} d & & & & & \\ 1 & d-2 & & & 0 & \\ & 2 & d-4 & & & \\ & & \ddots & \ddots & & \\ & 0 & & & 2-d & \\ & & & & d & -d \end{pmatrix}$$

$$A^* = \begin{pmatrix} d & \varphi_1 & & & 0 & \\ & d-2 & \varphi_2 & & & \\ & & d-4 & \ddots & & \\ & & & \ddots & & \\ & 0 & & & 2-d & \varphi_d \\ & & & & & -d \end{pmatrix} \qquad \varphi_i = 2(i-d-1)$$

CRS $-d$

Then A, A^* is a TD pair (of Krawtchouk type)

Reason:

If we diagonalize A then A^* becomes tridiagonal ...
A^* .. A

- - -

Here

$$\dim V_i = 1 = \dim V_i^* \qquad 0 \leq i \leq d$$

"Leonard pair"

Connection to sl_2

Write
$$A = H + F, \qquad A^* = H - 2E$$

where
$$H = \begin{pmatrix} d & & & 0 \\ & d-2 & & \\ & & \ddots & \\ 0 & & & -d \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & d & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & d & & & 0 \\ & 0 & \ddots & & \\ & & \ddots & 2 & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}$$

One checks
$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H$$
where $[r, s] = rs - sr$

So H, E, F give a rep of Lie algebra sl_2

One checks
$$[A, [A, A^*]] = 4A^*$$
$$[A^*, [A^*, A]] = 4A$$

Turns out any TD pair satisfy a similar pair of relations, called the TD relations

Case	TD relations
Krawtchouk	$[A,[A,[A,A^*]]] = 4[A,A^*]$ $[A^*,[A^*,[A^*,A]]] = 4[A^*,A]$ "Dolan/Grady rels" (defining rels for Onsager Lie alg)
q-geom	$[A,[A,[A,A^*]_q]_1]_{q^{-1}} = 0$ $[A^*,[A^*,[A^*,A]_q]_1]_{q^{-1}} = 0$ $[r,s]_q = qrs - q^{-1}sr$ "q-Serre rels" (defining rels for positive part U_q^+ of $U_q\hat{sl}_2$)
q-Racah	$[A,[A,[A,A^*]_q]_1]_{q^{-1}} = (q^2-q^{-2})^2[A^*,A]$ $[A^*,[A^*,[A^*,A]_q]_1]_{q^{-1}} = (q^2-q^{-2})^2[A,A^*]$ "q-Dolan/Grady" (defining rels for q-Onsager alg)

TD pairs and quantum groups

Next goal:

Using a TD pair A, A^* on V of q-geometric type,

turn V into a $U_q \hat{sl}_2$-module

Recall A, A^* satisfy the q-Serre rels

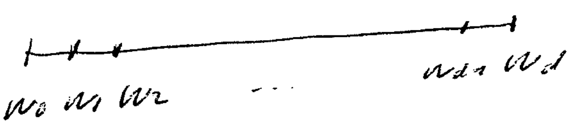
then eigenvalues are

$$\theta_i = q^{d-2i} = \theta_i^* \qquad 0 \le i \le d$$

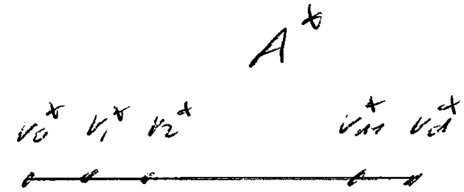
DEF A <u>decomposition of V</u> is a sequence $\{W_i\}_{i=0}^{d}$

of nonzero subspaces whose direct sum is V

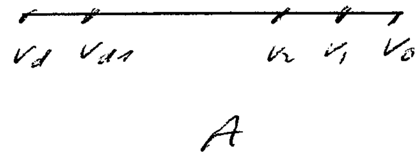
Draw



or just



Eigenspace decomps:



Define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}) \qquad 0 \le i \le d$$

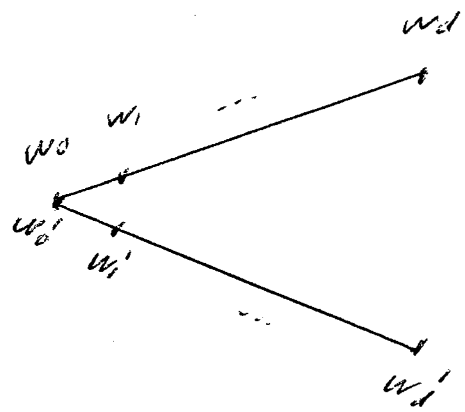
$\{U_i\}_{i=0}^{d}$ is a decomp of V "Split decomp"

By constr

$$U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^* \qquad 0 \le i \le d$$

$$U_i + U_{i+1} + \cdots + U_d = V_0 + V_1 + \cdots + V_{d-i}$$

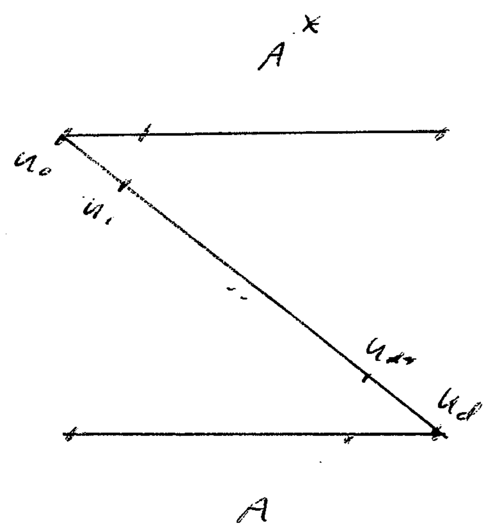
<u>Notation</u> Given decomps $\{w_i\}_{i=0}^{d}$, $\{w_i'\}_{i=0}^{d}$ of V



means

$$w_0+w_1+\cdots+w_i = w_0'+w_1'+\cdots+w_i' \qquad 0\le i\le d$$

So



LEM For $0 \le i \le d$

$A u_i \subseteq u_i + u_{i+1}$

$A^* u_i \subseteq u_i + u_{i-1}$ $(u_{-1} = 0)$

pf -

Define $K \in End(V)$ st

$$(K - q^{d-2i} I) u_i = 0 \qquad 0 \le i \le d$$

By above LEM,

$$\frac{qKA - q^{-1}AK}{q - q^{-1}} = I,$$

$$\frac{qK^{-1}A^* - q^{-1}A^*K^{-1}}{q - q^{-1}} = I.$$

We now give a more comprehensive result

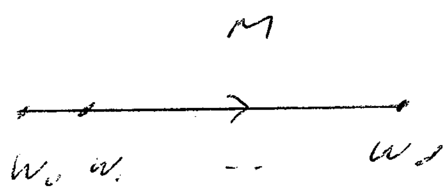
For any decomp $\{W_i\}_{i=0}^{d}$ of V, define

$M \in \mathrm{End}(V)$ s.t.

$$(M - q^{d-2i} I)\, W_i = 0 \qquad 0 \le i \le d$$

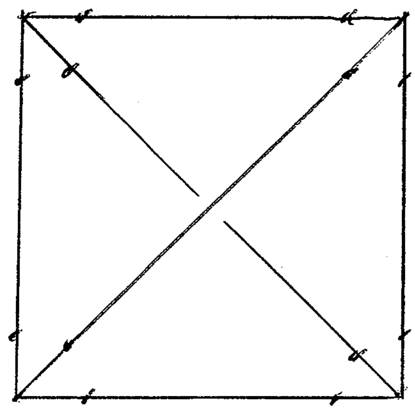
"M is induced by $\{W_i\}_{i=0}^{d}$"

Draw



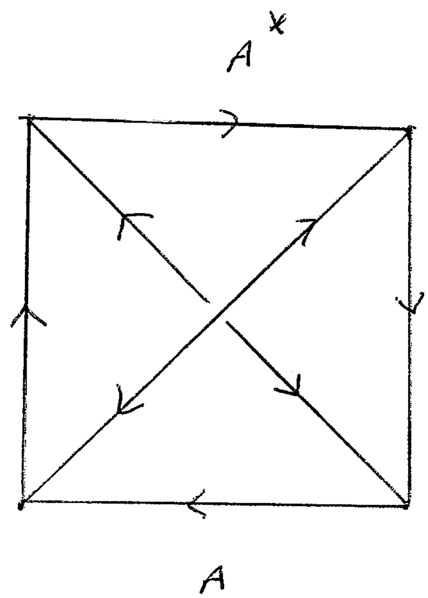
Prop $\exists$ decomps

A^*



A

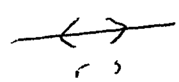
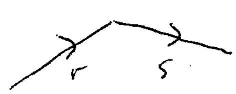
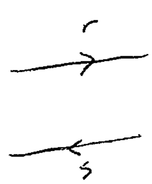
thm We have



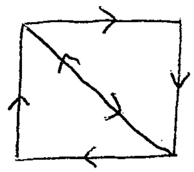
where:

each directed arc represents the lin trans in $End(V)$ induced by underlying decomp

Key

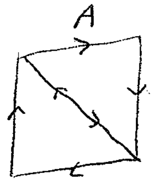
	means
	$rs = I$
	$\frac{qrs - q^{-1}sr}{q - q^{-1}} = I$
	r, s satisfy q-Serre rels

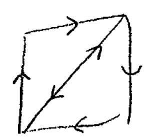
DEF The algebra $U_q\widehat{sl_2}$ has a presentation by gens + rels:



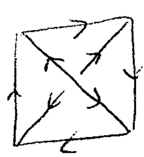
Each directed arc is a generator.
Relations are from Key.

COR Our TD pair A, A^* on V of q-geom type turn V into a $U_q\widehat{sl_2}$-module in two ways:





Note The q-tetrahedron algebra $\boxtimes_q$ has a pres



The algebra $U_q(sl_2)$ has a pres

"equitable presentation"

$\square$