Tridiagonal pairs in algebraic graph theory

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Let X denote a nonempty finite set.

$\text{Mat}_X(\mathbb{C})$ denotes the $\mathbb{C}$-algebra consisting of the matrices over $\mathbb{C}$ that have rows and columns indexed by X.

$V = \mathbb{C}X$ denotes the vector space over $\mathbb{C}$ consisting of column vectors with rows indexed by X.

$\text{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

Endow V with a Hermitean inner product

$$\langle u, v \rangle = u^t \overline{v} \qquad\qquad (u, v \in V)$$

For each $x \in X$ let $\hat{x}$ denote the vector in V that has a 1 in coordinate x and 0 in all other coordinates.

Observe $\{\hat{x} | x \in X\}$ is an orthonormal basis for V.

The graph Γ

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, and path-length distance function ∂.

For an integer $i \geq 0$ and $x \in X$ let

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

The graphs of interest

Our main case of interest is when Γ is “highly regular” in a certain way.

A good example to keep in mind is the D-**dimensional hypercube**, also called the **binary Hamming graph** $H(D, 2)$.

Note that $H(2, 2)$ is a 4-cycle; this will be used as a running example.

The adjacency matrix

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the $(0,1)$-adjacency matrix of Γ.

For $x \in X$,

$$A\hat{x} = \sum_{y \in \Gamma(x)} \hat{y}$$

Example

For $H(2,2)$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The adjacency algebra M

Let M denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by A.

M is called the **adjacency algebra** of Γ.

M is commutative and semisimple.

Example

For $H(2,2)$, M has a basis

$$I, A, J$$

where the matrix J has every entry 1.

The primitive idempotents of Γ

Since M is semisimple it has basis $\{E_i\}_{i=0}^{d}$ such that

$$\begin{aligned} E_i E_j &= \delta_{ij} E_i \qquad\qquad (0 \le i, j \le d), \\ I &= \sum_{i=0}^{d} E_i. \end{aligned}$$

We call $\{E_i\}_{i=0}^{d}$ the **primitive idempotents** of Γ.
Write

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

For $0 \le i \le d$ the scalar θ_i is the **eigenvalue** of A for E_i.

The primitive idempotents, cont.

Example

For $H(2,2)$ we have $\theta_0 = 2$, $\theta_1 = 0$, $\theta_2 = -2$. Moreover

$$E_0 = 1/4J,$$

$$E_1 = 1/4 \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix},$$

$$E_2 = 1/4 \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The eigenspaces of A

The vector space V decomposes as

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum)}$$

For $0 \leq i \leq d$ the space $E_i V$ is the **eigenspace of** A associated with the eigenvalue θ_i.

the matrix E_i represents the **orthogonal projection** onto $E_i V$.

The dual primitive idempotents of Γ

Until further notice

$$\textbf{fix } x \in X.$$

We call x the **base vertex**.

Define $D = D(x)$ by

$$D = \max\{\partial(x, y) \mid y \in X\}$$

We call D the **diameter of Γ with respect to** x.

For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i. \end{cases} \qquad (y \in X).$$

The dual primitive idempotents, cont.

Example

For $H(2,2)$,

$$E_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_1^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual primitive idempotents, cont.

The $\{E_i^*\}_{i=0}^D$ satisfy

$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i, j \leq D),$$
$$I = \sum_{i=0}^{D} E_i^*.$$

We call $\{E_i^*\}_{i=0}^D$ the **dual primitive idempotents of Γ with respect to** x.

The dual adjacency algebra

The $\{E_i^*\}_{i=0}^D$ form a basis for a semisimple commutative subalgebra of $\text{Mat}_X(\mathbb{C})$ denoted $M^* = M^*(x)$.

We call M^* the **dual adjacency algebra of Γ with respect to x.**

The subconstituents of Γ

The vector space V decomposes as

$$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(orthogonal direct sum)}$$

The above summands are the common eigenspaces for M^*.

These eigenspaces have the following combinatorial interpretation. For $0 \leq i \leq D$,

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in \Gamma_i(x)\}$$

We call $E_i^* V$ the **ith subconstituent of Γ with respect to x**.

The matrix E_i^* represents the **orthogonal projection** onto $E_i^* V$.

The subconstituent algebra T

So far we defined the adjacency algebra M and the dual adjacency algebra M^*. We now combine M and M^* to get a larger algebra.

Definition

(Ter 92) Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. T is called the **subconstituent algebra of Γ with respect to x**.

T is finite-dimensional.

T is noncommutative in general.

T is semisimple

T is semi-simple because it is closed under the conjugate-transpose map.

So by the Wedderburn theory the algebra T is isomorphic to a direct sum of matrix algebras.

Example

For $H(2,2)$,

$$T \simeq \text{Mat}_3(\mathbb{C}) \oplus \mathbb{C}.$$

Moreover $\dim(T) = 10$.

T is semisimple, cont.

Example

(Junie Go 2002) For $H(D,2)$,

$$T \simeq \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-3}(\mathbb{C}) \oplus \cdots$$

Moreover $\dim(T) = (D+1)(D+2)(D+3)/6$.

T-modules

We mentioned that T is closed under the conjugate-transpose map.

So for each T-module $W \subseteq V$, its orthogonal complement $W^{\perp}$ is also a T-module.

Therefore V decomposes into an orthogonal direct sum of irreducible T-modules.

Problem

(i) How does the above decomposition reflect the combinatorial properties of Γ? (ii) For which graphs Γ is the above decomposition particularly nice?

The dual adjacency matrix

We now describe a family of graphs for which the irreducible T-modules are nice.

These graphs possess a certain matrix called a **dual adjacency matrix**.

To motivate this concept we consider some relations in T.

By the triangle inequality

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

This is reformulated as follows.

Lemma

For $0 \leq i, j \leq D$,

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j] > 1.$$

The dual adjacency matrix

Definition

Referring to the graph Γ, consider a matrix $A^* \in \text{Mat}_X(\mathbb{C})$ that satisfies both conditions below:

(i) A^* generates M^*;

(ii) For $0 \le i, j \le d$,

$$E_i A^* E_j = 0 \quad \text{if } |i - j] > 1.$$

We call A^* a **dual adjacency matrix** (with respect to x and the given ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents).

A dual adjacency matrix A^* is **diagonal**.

The dual adjacency matrix

A dual adjacency matrix A^* acts on the eigenspaces of A as follows.

$$A^* E_i V \subseteq E_{i-1}V + E_i V + E_{i+1}V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

An example

Example

$H(2,2)$ has a dual adjacency matrix

$$A^* = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

Example

$H(D,2)$ has a dual adjacency matrix

$$A^* = \sum_{i=0}^{D} (D-2i)E_i^*.$$

More examples

The following graphs have a dual adjacency matrix.

Any **strongly-regular** graph.

Any Q-**polynomial distance-regular** graph, for instance:

- cycle
- Hamming graph
- Johnson graph
- Grassman graph
- Dual polar spaces
- Bilinear forms graph
- Alternating forms graph
- Hermitean forms graph
- Quadratic forms graph

See the book **Distance-Regular Graphs** by Brouwer, Cohen, Neumaier.

How A and A^* are related

To summarize so far, for our graph Γ the adjacency matrix A and any dual adjacency matrix A^* generate T. Moreover they act on each other's eigenspaces in the following way:

$$AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \qquad (0 \le i \le D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

$$A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \le i \le d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

To clarify this situation we reformulate it as a problem in linear algebra.

We now define a linear-algebraic object called a **TD pair**.

From now on $\mathbb{F}$ denotes a field.

V will denote a vector space over $\mathbb{F}$ with finite positive dimension.

We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$.

Definition of a Tridiagonal pair (Ito, Tanabe, Ter 2001)

We say the pair A, A^* is a **TD pair** on V whenever (1)–(4) hold below.

1. Each of A, A^* is diagonalizable on V.
2. There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that
$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$
where $V_{-1} = 0$, $V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^D$ of the eigenspaces of A^* such that
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq D),$$
where $V_{-1}^* = 0$, $V_{D+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^* W \subseteq W$ and $W \neq 0$ and $W \neq V$.

The diameter

Referring to our definition of a TD pair,

it turns out $d = D$; we call this common value the **diameter** of the pair.

Each irreducible T-module gives a TD pair

Briefly returning to the graph Γ, the adjacency matrix and any dual adjacency matrix act on each irreducible T-module as a TD pair.

This motivates us to understand TD pairs.

Special case: Leonard pairs

In our study of TD pairs we begin with a special case called a **Leonard pair**.

A Leonard pair is a TD pair for which the eigenspaces V_i and V_i^* all have dimension 1. These are classified up to isomorphism (Ter 2001).

To describe the Leonard pairs, we recall some notation. A square matrix is called **tridiagonal** whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

A tridiagonal matrix is **irreducible** whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Lemma

A pair of linear transformations $A : V \to V$ and $A^* : V \to V$ is a Leonard pair if and only if:

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
2. There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Example of a Leonard pair

For any integer $d \geq 0$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix},$$

$$A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than d.

Reason: There exists an invertible matrix P such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q-Racah,
q-Hahn,
dual q-Hahn,
q-Krawtchouk,
dual q-Krawtchouk,
quantum q-Krawtchouk,
affine q-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans (char($\mathbb{F}$) $= 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

Recommended reading

The theory of Leonard pairs is summarized in

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; `arXiv:math.QA/0408390`.

We now return to general TD pairs.

In 2010 my collaborators Tatsuro Ito, Kazumasa Nomura and I classified up to isomorphism the TD pairs over an algebraically closed field.

To be precise, we classified up to isomorphism a more general family of TD pairs said to be **sharp**.

I will describe this result shortly.

When working with a TD pair, it is helpful to consider a closely related object called a **TD system**.

We will define a TD system over the next few slides.

Standard orderings

Referring to our definition of a TD pair,

An ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

In this case, the ordering $\{V_{d-i}\}_{i=0}^{d}$ is also standard and no further ordering is standard.

A similar discussion applies to A^*.

Primitive idempotents

Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** E is the projection onto that eigenspace.

In other words $E - I$ vanishes on the eigenspace and E vanishes on all the other eigenspaces.

TD systems

Definition

By a **TD system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

that satisfies the following:

1. A, A^* is a TD pair on V.
2. $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A.
3. $\{E_i^*\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A^*.

Until further notice we fix a TD system Φ as above.

The eigenvalues

For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with the eigenspace E_iV (resp. E^*_iV).

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

A three-term recurrence

Theorem (Ito+Tanabe+T, 2001)

The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i *for* $2 \leq i \leq d-1$.

Let $\beta + 1$ denote the common value of the above expressions.

Solving the recurrence

For the above recurrence the "simplest" solution is

$$\theta_i = d - 2i \ \ (0 \le i \le d),$$
$$\theta^*_i = d - 2i \ (0 \le i \le d).$$

In this case $\beta = 2$.

For this solution our TD system Φ is said to have **Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence another solution is

$$\theta_i = q^{d-2i} \quad (0 \leq i \leq d),$$
$$\theta^*_i = q^{d-2i} \; (0 \leq i \leq d),$$
$$q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1.$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Krawtchouk type**.

For the above recurrence the "most general" solution is

$$\begin{aligned}
&\theta_i = a + bq^{2i-d} + cq^{d-2i} \ \ (0 \le i \le d), \\
&\theta^*_i = a^* + b^*q^{2i-d} + c^*q^{d-2i} \ (0 \le i \le d), \\
&q, \ a, \ b, \ c, \ a^*, \ b^*, \ c^* \in \overline{\mathbb{F}}, \\
&q \neq 0, \ \ q^2 \neq 1, \ \ q^2 \neq -1, \ \ bb^*cc^* \neq 0.
\end{aligned}$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Racah type**.

Some notation

For later use we define some polynomials in an indeterminate λ.

For $0 \leq i \leq d$,

$$
\begin{aligned}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}), \\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{i-1}^*), \\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*)\cdots(\lambda - \theta_{d-i+1}^*).
\end{aligned}
$$

Note that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

The shape

It is known that for $0 \leq i \leq d$ the eigenspaces $E_i V$, $E_i^* V$ have the same dimension; we denote this common dimension by ρ_i.

Lemma (Ito+Tanabe+T, 2001)

The sequence $\{\rho_i\}_{i=0}^d$ is **symmetric** *and* **unimodal**; *that is*

$$\rho_i = \rho_{d-i} \qquad (0 \leq i \leq d),$$
$$\rho_{i-1} \leq \rho_i \qquad (1 \leq i \leq d/2).$$

We call the sequence $\{\rho_i\}_{i=0}^d$ the **shape** of Φ.

A bound on the shape

Theorem (Ito+Nomura+Ter, 2009)

The shape $\{\rho_i\}_{i=0}^d$ *of* Φ *satisfies*

$$\rho_i \leq \rho_0 \binom{d}{i} \qquad (0 \leq i \leq d).$$

The parameter ρ_0

What are the possible values for ρ_0?

The answer depends on the precise nature of the field $\mathbb{F}$.

We will explain this after a few slides.

Some relations

Lemma

Our TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *satisfies the following relations:*

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^* \qquad 0 \le i,j \le d,$$

$$1 = \sum_{i=0}^d E_i, \qquad 1 = \sum_{i=0}^d E_i^*,$$

$$A = \sum_{i=0}^d \theta_i E_i, \qquad A^* = \sum_{i=0}^d \theta_i^* E_i^*,$$

$$E_i^* A^k E_j^* = 0 \quad \text{if} \;\; k < |i-j| \qquad 0 \le i,j,k \le d,$$

$$E_i A^{*k} E_j = 0 \quad \text{if} \;\; k < |i-j| \qquad 0 \le i,j,k \le d.$$

We call these last two equations the **triple product relations**.

The algebra T

Given the relations on the previous slide, it is natural to consider the subalgebra of $\mathrm{End(V)}$ generated by A; A^*; $\{E_i\}_{i=0}^d$; $\{E_i^*\}_{i=0}^d$.

We call this algebra T.

Consider the space $E_0^* T E_0^*$.

Observe that $E_0^* T E_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity E_0^*.

The algebra $E_0^* T E_0^*$

Theorem (Ito+Nomura+T, 2007)

(i) The $\mathbb{F}$-algebra $E_0^* T E_0^*$ is commutative and generated by

$$E_0^* A^i E_0^* \qquad 1 \leq i \leq d.$$

(ii) $E_0^* T E_0^*$ is a field.
(iii) Viewing this field as a field extension of $\mathbb{F}$, the index is ρ_0.

The parameter ρ_0

Corollary (Ito+Nomura+Ter, 2007)

If $\mathbb{F}$ is algebraically closed then $\rho_0 = 1$.

We now consider some more relations in T.

The tridiagonal relations

Theorem (Ito+Tanabe+T, 2001)

For our TD system Φ *there exist scalars* $\gamma, \gamma^*, \varrho, \varrho^*$ *in* $\mathbb{F}$ *such that*

$$\begin{aligned} A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ = \quad \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ = \quad \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

The above equations are called the **tridiagonal relations**.

The Dolan-Grady relations

In the Krawtchouk case the tridiagonal relations become the **Dolan-Grady relations**

$$[A,[A,[A,A^*]]] = 4[A,A^*],$$

$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A].$$

Here $[r,s] = rs - sr$.

The Dolan Grady relations are the defining relations for the **Onsager Algebra**.

The q-Serre relations

In the q-Krawtchouk case the tridiagonal relations become the cubic q-**Serre relations**

$$
\begin{aligned}
A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 &= 0, \\
A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} &= 0.
\end{aligned}
$$

$$
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots
$$

The q-Serre relations are among the defining relations for the **quantum group** $U_q(\widehat{\mathfrak{sl}_2})$.

The sharp case

At this point it is convenient to make an assumption about our TD system Φ.

Φ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^{d}$ is the shape of Φ.

If the ground field $\mathbb{F}$ is algebraically closed then Φ is sharp.

Until further notice assume Φ is sharp.

For $0 \le i \le d$ define

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V).$$

It is known that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

and for $0 \le i \le d$ both

$$\begin{aligned} U_0 + \cdots + U_i &= E_0^* V + \cdots + E_i^* V, \\ U_i + \cdots + U_d &= E_i V + \cdots + E_d V. \end{aligned}$$

We call the sequence $\{U_i\}_{i=0}^d$ the **split decomposition** of V with respect to Φ.

The split decomposition, cont.

Theorem (Ito+Tanabe+T, 2001)

For $0 \leq i \leq d$ *both*

$$\begin{aligned} (A - \theta_i I)U_i &\subseteq U_{i+1}, \\ (A^* - \theta^*_i I)U_i &\subseteq U_{i-1}, \end{aligned}$$

where $U_{-1} = 0$, $U_{d+1} = 0$.

Observe that for $0 \leq i \leq d$,

$$\begin{aligned}(A-\theta_{i-1}I)\cdots(A-\theta_1 I)(A-\theta_0 I)U_0 &\subseteq U_i,\\ (A^*-\theta^*_1 I)\cdots(A^*-\theta^*_{i-1}I)(A^*-\theta^*_i I)U_i &\subseteq U_0.\end{aligned}$$

Therefore U_0 is invariant under

$$(A^*-\theta^*_1 I)\cdots(A^*-\theta^*_i I)(A-\theta_{i-1}I)\cdots(A-\theta_0 I).$$

Let ζ_i denote the corresponding eigenvalue and note that $\zeta_0 = 1$.

We call the sequence $\{\zeta_i\}_{i=0}^{d}$ the **split sequence** of Φ.

The split sequence $\{\zeta_i\}_{i=0}^{d}$ is characterized as follows.

Lemma (Nomura+T, 2007)

For $0 \leq i \leq d$,

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

A restriction on the split sequence

The split sequence $\{\zeta_i\}_{i=0}^{d}$ satisfies two inequalities.

Lemma (Ito+Tanabe+T, 2001)

$$
\begin{aligned}
0 &\neq E_0^* E_d E_0^*, \\
0 &\neq E_0^* E_0 E_0^*.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
0 &\neq \zeta_d, \\
0 &\neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.
\end{aligned}
$$

The parameter array

Lemma (Ito+ Nomura+T, 2008)

The TD system Φ is determined up to isomorphism by the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d).$$

We call this sequence the **parameter array** of Φ.

A classification of sharp tridiagonal systems

We are now ready to state our classification theorem for sharp TD systems. The idea is the following.

We have seen that each sharp TD system Φ is attached to a unique parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d). \tag{1}$$

By the previous slide, this parameter array uniquely determins Φ up to isomorphism.

Therefore, to classify the sharp TD systems up to isomorphism, it suffices to specify which sequences (1) can be their parameter array.

This is done on the next slide.

A classification of sharp tridiagonal systems

Theorem (Ito+Nomura+Ter, 2009)

Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ (1) *denote a sequence of scalars in* $\mathbb{F}$. *Then there exists a sharp TD system* Φ *over* $\mathbb{F}$ *with parameter array (1) if and only if:*

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i,j \leq d)$;

(ii) *the expressions* $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$, $\frac{\theta_{i-2}^*-\theta_{i+1}^*}{\theta_{i-1}^*-\theta_i^*}$ *are equal and independent of* i *for* $2 \leq i \leq d-1$;

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i.$$

Suppose (i)–(iii) *hold. Then* Φ *is unique up to isomorphism of TD systems.*

The classification: proof summary

In the proof the hard part is to construct a TD system with given parameter array of q-Racah type.

To do this we make use of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$.

Using the parameter array we identify two elements in $U_q(\widehat{\mathfrak{sl}_2})$ that satisfy some tridiagonal relations.

We let these elements act on a certain $U_q(\widehat{\mathfrak{sl}_2})$-module of the form $W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where each W_i is an evaluation module of dimension 2.

This action yields the desired TD system after a reduction process.

Connections

Recently TD pairs have appeared in the study of **statistical mechanics**, in connection with the **open XXZ spin chain**. Some references are

P. Baseilhac, K. Koizumi. A new (in)finite dimensional algebra for quantum integrable models arXiv:math-ph/0503036

P. Baseilhac. A family of tridiagonal pairs and related symmetric functions. arXiv:math-ph/0604035

P. Baseilhac, K. Koizumi. A deformed analogue of Onsager's symmetry in the XXZ open spin chain. arXiv:hep-th/0507053

P. Baseilhac, K. Koizumi. Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory. arXiv:hep-th/0703106

More connections

Recently the tridiagonal relations have appeared in the study of **quantum symmetric pairs**:

Stefan Kolb. Quantum symmetric Kac-Moody pairs arXiv:1207.6036.

A variation on the tridiagonal relations appears in **knot theory**:

Doug Bullock, Jozef H. Przytycki. Multiplicative structure of Kauffman bracket skein module quantizations Proceedings of the AMS (1999), 923–931.

A variation on the subconstituent algebra has appeared in the study of **quantum probability**:

Akihiro Hora, Nobuaki Obata. Quantum Probability and Spectral Analysis of Graphs Series: Theoretical and Mathematical Physics 2007.

Summary

In Part I we discussed the **subconstituent algebra** T of a graph Γ and considered how the standard module V decomposes into a direct sum of irreducible T-modules.

We identified a class of graphs for which the irreducible T-modules are nice; these graphs possess a **dual adjacency matrix**.

For these graphs the adjacency matrix and dual adjacency matrix act on each irreducible T-module as a **TD pair**.

In Part II we considered general TD pairs. We discussed the **eigenvalues, dual eigenvalues, shape, tridiagonal relations**, and **parameter array**.

We then classified up to isomorphism the TD pairs over an algebraically closed field. In the future we hope to apply this classification to the study of graphs.

Thank you for your attention!

THE END