Totally Bipartite Tridiagonal Pairs

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Contents

We now define a tridiagonal pair.

Let $\mathbb{F}$ denote a field.

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension.

Consider two $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \le i \le d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \le i \le \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

History and connections

The tridiagonal pairs were introduced in 1999 by Ito, Tanabe, and Terwilliger.

The tridiagonal pairs over an algebraically closed field were classified up to isomorphism by Ito, Nomura, and Terwilliger (2011).

Tridiagonal pairs are related to:

- Q-polynomial distance-regular graphs,
- the orthogonal polynomials of the Askey scheme,
- the Askey-Wilson, Onsager, and q-Onsager algebras,
- the double affine Hecke algebra of type $(C_1^\vee, C_1)$,
- the Lie algebras $\mathfrak{sl}_2$ and $\widehat{\mathfrak{sl}}_2$,
- the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{\mathfrak{sl}}_2)$,
- integrable models in statistical mechanics,
- the Skein algebra for knots and links.

Totally bipartite tridiagonal pairs

In the study of tridiagonal pairs, it is helpful to begin with a special case, said to be **Totally Bipartite** (or **TB**).

This case is easier to handle, but sufficiently rich to hint at what happens in general.

Our goal in this talk is to describe the TB tridiagonal pairs, starting from first principles and without invoking results about general tridiagonal pairs.

We acknowledge the considerable prior work on the TB case, due to **George Brown, Brian Curtin, Sougang Gao, Miloslav Havlicek, Bo Hou, Anatoliy Klimyk, Severin Posta.**

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension.

Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps from V to V.

The definition of a TB tridiagonal pair

A **TB tridiagonal pair on** V is an ordered pair A, A^* of elements in $\mathrm{End}(V)$ such that:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_{i+1} \qquad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Comments

According to a common notational convention, A^* denotes the conjugate-transpose of A.

We are not using this convention.

In a TB tridiagonal pair, the elements A and A^* are arbitrary subject to (i)–(iv) above.

If A, A^* is a TB tridiagonal pair on V, then so is A^*, A.

If A, A^* is a TB tridiagonal pair on V, then so is hA, h^*A^* for all nonzero $h, h^* \in \mathbb{F}$.

Assume that $\dim(V) = 1$ and $A = 0$, $A^* = 0$. Then A, A^* is a TB tridiagonal pair on V, said to be **trivial**.

Isomorphisms

Let A, A^* denote a TB tridiagonal pair on V.

Let $A', A^{*\prime}$ denote a TB tridiagonal pair on V'.

By an **isomorphism of TB tridiagonal pairs from A, A^* to $A', A^{*\prime}$**, we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ such that $\sigma A = A' \sigma$ and $\sigma A^* = A^{*\prime} \sigma$.

We say that A, A^* and $A', A^{*\prime}$ are **isomorphic** whenever there exists an isomorphism of TB tridiagonal pairs from A, A^* to $A', A^{*\prime}$.

Standard orderings

Let A, A^* denote a TB tridiagonal pair on V. An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever it satisfies (1).

Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A.

Then the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is standard, and no further ordering is standard.

Similar comments apply to A^*.

TB tridiagonal systems

In our study of TB tridiagonal pairs, it is useful to introduce a related concept called a **TB tridiagonal system**.

To define this, we first recall the notion of a primitive idempotent.

Given an eigenspace W of a diagonalizable linear transformation, the corresponding **primitive idempotent** acts on W as the identity map, and acts on the other eigenspaces as zero.

The definition of a TB tridiagonal system

Definition

By a **TB tridiagonal system on** V, we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

such that

(i) A, A^* is a TB tridiagonal pair on V;

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^{\delta}$ is a standard ordering of the primitive idempotents of A^*.

The D_4 action

Consider a TB tridiagonal system $\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$ on V.

Each of the following is a TB tridiagonal system on V:

$$\Phi^* = (A^*, \{E_i^*\}_{i=0}^{\delta}, A, \{E_i\}_{i=0}^{d});$$
$$\Phi^{\downarrow} = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_{\delta-i}^*\}_{i=0}^{\delta});$$
$$\Phi^{\Downarrow} = (A, \{E_{d-i}\}_{i=0}^{d}, A^*, \{E_{\delta}^*\}_{i=0}^{\delta}).$$

The D_4 action, cont.

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all TB tridiagonal systems,

$$*^2 = 1, \qquad \downarrow^2 = 1, \qquad \Downarrow^2 = 1, \tag{2}$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow \Downarrow = \downarrow \Downarrow . \tag{3}$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the relations (2), (3) is the dihedral group D_4. Recall that D_4 is the group of symmetries of a square, and has 8 elements.

The elements $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all TB tridiagonal systems.

Isomorphisms for TB tridiagonal systems

Earlier we defined the isomorphism concept for TB tridiagonal pairs.

The isomorphism concept for TB tridiagonal systems is similarly defined.

Until further notice, let

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

denote a TB tridiagonal system on V.

The triple product relations

Lemma

With the above notation,

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| \neq 1; \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i, j \leq \delta),$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| \neq 1; \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i, j \leq d).$$

The eigenvalues

Definition

For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A corresponding to E_i. For $0 \leq i \leq \delta$ let θ^*_i denote the eigenvalue of A^* corresponding to E^*_i.

Definition

We call $\{\theta_i\}_{i=0}^{d}$ (resp. $\{\theta^*_i\}_{i=0}^{\delta}$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ. We call $(\{\theta_i\}_{i=0}^{d}, \{\theta^*_i\}_{i=0}^{\delta})$ the **eigenvalue array** of Φ.

The eigenspaces

Lemma

We have $d = \delta$ *and*

$$\dim E_i V = 1, \qquad \dim E_i^* V = 1 \qquad (0 \leq i \leq d).$$

Moreover $\dim V = d + 1$. *We call* d *the* **diameter** *of* Φ.

The Φ-standard basis for V

Lemma

There exists a basis $\{v_i\}_{i=0}^d$ for V such that:

(i) $v_i \in E_i^* V$ for $0 \leq i \leq d$;

(ii) $\sum_{i=0}^d v_i \in E_0 V$.

Such a basis is said to be **Φ-standard**.

Matrix representations

For $X \in \mathrm{End}(V)$ let $X^\natural$ denote the matrix that represents X with respect to a Φ-standard basis.

For instance

$$(A^*)^\natural = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Matrix representations, cont.

Lemma

We have

$$A^{\natural} = \begin{pmatrix} 0 & b_0 & & & & \mathbf{0} \\ c_1 & 0 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & 0 \end{pmatrix},$$

where

$$c_i = \frac{\theta_1\theta_i^* - \theta_0\theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \qquad (1 \le i \le d-1), \qquad c_d = \theta_0,$$

$$b_i = \frac{\theta_1\theta_i^* - \theta_0\theta_{i-1}^*}{\theta_{i+1}^* - \theta_{i-1}^*} \qquad (1 \le i \le d-1), \qquad b_0 = \theta_0.$$

Corollary

The TB tridiagonal system Φ *is determined up to isomorphism by its eigenvalue array* $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$.

The classification of TB tridiagonal systems

We now classify up to isomorphism the TB tridiagonal systems.

Fix an integer $d \geq 1$, and consider a sequence of scalars in $\mathbb{F}$:

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d). \tag{4}$$

Theorem

There exists a TB tridiagonal system Φ over $\mathbb{F}$ that has eigenvalue array (4) if and only if the following (i)–(iii) hold:

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) *there exists* $\beta \in \mathbb{F}$ *such that for* $1 \leq i \leq d-1$,

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0, \qquad \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0;$$

(iii) $\theta_i + \theta_{d-i} = 0$, $\theta_i^* + \theta_{d-i}^* = 0$ $(0 \leq i \leq d)$.

In this case Φ is unique up to isomorphism of TB tridiagonal systems. We call β the **fundamental parameter** *of Φ.*

Definition

Fix an integer $d \geq 1$, and consider a sequence of scalars in $\mathbb{F}$:

$$(\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d}).$$

This sequence is called an **eigenvalue array** whenever it satisfies the conditions (i)–(iii) in the previous theorem.

We call d the **diameter** of the array.

The eigenvalue array, cont.

By construction, for $d \geq 1$ the following sets are in bijection:

- the isomorphism classes of TB tridiagonal systems over $\mathbb{F}$ that have diameter d;
- the eigenvalue arrays over $\mathbb{F}$ that have diameter d.

Our next goal is to display the eigenvalue arrays in closed form.

For each array, we give the corresponding c_i, b_i and fundamental parameter β.

Fix an integer $d \geq 1$, and consider a sequence of scalars in $\mathbb{F}$:

$$(\{\theta_i\}_{i=0}^{d}; \{\theta^*_i\}_{i=0}^{d}). \tag{5}$$

As a warmup we handle the cases $d = 1$, $d = 2$ separately.

The eigenvalue arrays in closed form.

Lemma

For $d = 1$ the following are equivalent:

(i) *the sequence* (5) *is an eigenvalue array over* $\mathbb{F}$*;*

(ii) $\mathrm{Char}(\mathbb{F}) \neq 2$*, the scalars* θ_0*,* θ^*_0 *are nonzero, and* $\theta_1 = -\theta_0$*,* $\theta^*_1 = -\theta^*_0$*.*

Assume that (i), (ii) *hold. Then*

$$c_1 = \theta_0, \qquad b_0 = \theta_0.$$

Moreover $\beta \in \mathbb{F}$*.*

Lemma

For $d = 2$ the following are equivalent:

(i) *the sequence* (5) *is an eigenvalue array over* $\mathbb{F}$*;*

(ii) $\mathrm{Char}(\mathbb{F}) \neq 2$*, the scalars* θ_0, θ^*_0 *are nonzero, and*

$$\theta_1 = 0, \qquad \theta_2 = -\theta_0, \qquad \theta^*_1 = 0, \qquad \theta^*_2 = -\theta^*_0.$$

Assume that (i), (ii) *hold. Then*

$$c_1 = \theta_0/2, \qquad c_2 = \theta_0, \qquad b_0 = \theta_0, \qquad b_1 = \theta_0/2.$$

Moreover $\beta \in \mathbb{F}$.

The eigenvalue arrays in closed form, cont.

Until further notice, assume that $d \geq 3$.

In the following examples we display all the eigenvalue arrays over $\mathbb{F}$ with diameter d.

The eigenvalue arrays in closed form, cont.

Example

Assume that $\mathrm{Char}(\mathbb{F})$ is 0 or greater than d. Assume that there exist nonzero $h, h^* \in \mathbb{F}$ such that

$$\theta_i = h(d-2i), \qquad \theta^*_i = h^*(d-2i) \qquad (0 \le i \le d).$$

Then (5) is an eigenvalue array over $\mathbb{F}$ with fundamental parameter $\beta = 2$. For this array,

$$\begin{aligned} c_i &= hi && (1 \le i \le d), \\ b_i &= h(d-i) && (0 \le i \le d-1). \end{aligned}$$

The eigenvalue arrays in closed form, cont.

Example

Assume that d is even. Assume that $\mathrm{Char}(\mathbb{F})$ is 0 or greater than d. Assume that there exist nonzero $h, h^* \in \mathbb{F}$ such that

$$\theta_i = h(d-2i)(-1)^i, \qquad \theta^*_i = h^*(d-2i)(-1)^i \qquad (0 \le i \le d).$$

Then (5) is an eigenvalue array over $\mathbb{F}$ with fundamental parameter $\beta = -2$. For this array,

$$\begin{aligned} c_i &= hi && (1 \le i \le d), \\ b_i &= h(d-i) && (0 \le i \le d-1). \end{aligned}$$

Example

Assume that d is even and $\text{Char}(\mathbb{F}) \neq 2$. Let q denote a nonzero scalar in the algebraic closure $\overline{\mathbb{F}}$ such that $q^2 + q^{-2} \in \mathbb{F}$ and

$$q^{2i} \neq 1 \quad (1 \leq i \leq d), \qquad q^{2i} \neq -1 \quad (1 \leq i \leq d-1).$$

Assume that there exist nonzero $h, h^* \in \mathbb{F}$ such that

$$\theta_i = \frac{h(q^{d-2i} - q^{2i-d})}{q^2 - q^{-2}}, \qquad \theta^*_i = \frac{h^*(q^{d-2i} - q^{2i-d})}{q^2 - q^{-2}}$$

for $0 \leq i \leq d$. Then (5) is an eigenvalue array over $\mathbb{F}$ with fundamental parameter $\beta = q^2 + q^{-2}$. Moreover $\beta \neq \pm 2$.

Example

.. For this array,

$$
\begin{aligned}
c_i &= \frac{h(q^{2i} - q^{-2i})}{(q^2 - q^{-2})(q^{d-2i} + q^{2i-d})} && (1 \le i \le d-1), \\
c_d &= \frac{h(q^d - q^{-d})}{q^2 - q^{-2}}, \\
b_i &= \frac{h(q^{2d-2i} - q^{2i-2d})}{(q^2 - q^{-2})(q^{d-2i} + q^{2i-d})} && (1 \le i \le d-1), \\
b_0 &= \frac{h(q^d - q^{-d})}{q^2 - q^{-2}}.
\end{aligned}
$$

Example

Assume that d is odd and $\mathrm{Char}(\mathbb{F}) \neq 2$. Let q denote a nonzero scalar in $\overline{\mathbb{F}}$ such that $q^2 + q^{-2} \in \mathbb{F}$ and

$$q^{2i} \neq 1 \quad (1 \leq i \leq d), \qquad q^{2i} \neq -1 \quad (1 \leq i \leq d-1).$$

Assume that there exist nonzero $h, h^* \in \mathbb{F}$ such that

$$\theta_i = \frac{h(q^{d-2i} - q^{2i-d})}{q - q^{-1}}, \qquad \theta^*_i = \frac{h^*(q^{d-2i} - q^{2i-d})}{q - q^{-1}}$$

for $0 \leq i \leq d$. Then (5) is an eigenvalue array over $\mathbb{F}$ with fundamental parameter $\beta = q^2 + q^{-2}$. Moreover $\beta \neq \pm 2$.

The eigenvalue arrays in closed form, cont.

Example

.. For this array,

$$
\begin{aligned}
c_i &= \frac{h(q^{2i} - q^{-2i})}{(q - q^{-1})(q^{d-2i} + q^{2i-d})} \qquad && (1 \le i \le d-1),\\
c_d &= \frac{h(q^{d} - q^{-d})}{q - q^{-1}},\\
b_i &= \frac{h(q^{2d-2i} - q^{2i-2d})}{(q - q^{-1})(q^{d-2i} + q^{2i-d})} \qquad && (1 \le i \le d-1),\\
b_0 &= \frac{h(q^{d} - q^{-d})}{q - q^{-1}}.
\end{aligned}
$$

The eigenvalue arrays in closed form, cont.

Theorem

Every eigenvalue array over $\mathbb{F}$ with diameter $d \geq 3$ is listed in exactly one of the previous examples.

We now describe two relations satisfied by a TB tridiagonal pair.

Theorem

Let A, A^* denote a TB tridiagonal pair over $\mathbb{F}$ with fundamental parameter β. Then there exist $\varrho, \varrho^* \in \mathbb{F}$ such that

$$A^2 A^* - \beta A A^* A + A^* A^2 = \varrho A^*,$$
$$A^{*2} A - \beta A^* A A^* + A A^{*2} = \varrho^* A.$$

The above equations are a special case of the **Askey-Wilson relations**.

The Askey-Wilson relations, cont.

Lemma

The scalars ϱ, ϱ^* in the previous theorem satisfy

$$\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2,$$
$$\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2}$$

for $1 \leq i \leq d$.

Symmetries

We now describe some symmetries afforded by a TB tridiagonal pair.

Theorem

Let A, A^* denote a TB tridiagonal pair. Then the following TB tridiagonal pairs are mutually isomorphic:

$$A, A^*; \qquad -A, A^*; \qquad A, -A^*; \qquad -A, -A^*.$$

Self-dual TB tridiagonal pairs

Definition

A TB tridiagonal pair A, A^* is said to be **self-dual** whenever A, A^* is isomorphic to A^*, A.

Lemma

Let A, A^* denote a TB tridiagonal pair over $\mathbb{F}$. Then there exists $0 \neq \xi \in \mathbb{F}$ such that the TB tridiagonal pair $\xi A, A^*$ is self-dual.

Self-dual TB tridiagonal pairs

Theorem

Let A, A^* denote a self-dual TB tridiagonal pair. Then the following TB tridiagonal pairs are mutually isomorphic:

$$
\begin{array}{llll}
A, A^*; & -A, A^*; & A, -A^*; & -A, -A^*; \\
A^*, A; & -A^*, A; & A^*, -A; & -A^*, -A.
\end{array}
$$

We recall the notion of antiautomorphism.

Let $\mathcal{A}$ denote an $\mathbb{F}$-algebra.

By an **antiautomorphism** of $\mathcal{A}$, we mean an $\mathbb{F}$-linear bijection $\sigma : \mathcal{A} \to \mathcal{A}$ such that $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$.

Theorem

Let A, A^* denote a TB tridiagonal pair on V. Then there exists an antiautomorphism $\dagger$ of $\mathrm{End}(V)$ that fixes each of A, A^*. Moreover $(X^\dagger)^\dagger = X$ for all $X \in \mathrm{End}(V)$.

We will discuss additional symmetries shortly.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

From now on, assume that $\mathbb{F}$ is algebraically closed.

Let A, A^* denote a TB tridiagonal pair on V, with fundamental parameter β and diameter $d \geq 2$.

For convenience, we adjust our notation as follows:

From now on we abbreviate $B = A^*$.

Our next goal is to put the Askey-Wilson relations in a form said to be $\mathbb{Z}_3$-**symmetric**.

This is done by introducing a third element C.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

Definition

Let z, z', z'' denote scalars in $\mathbb{F}$ that satisfy

Case	$z'z''$	$z''z$
$\beta = 2$	$-\varrho$	$-\varrho^*$
$\beta = -2$	ϱ	ϱ^*
$\beta \neq \pm 2$	$\varrho(4-\beta^2)^{-1}$	$\varrho^*(4-\beta^2)^{-1}$

The above scalars ϱ, ϱ^* are from the Askey-Wilson relations.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

Theorem

Assume that $\beta = 2$. *Then there exists* $C \in \mathrm{End}(V)$ *such that*

$$BC - CB = zA,$$
$$CA - AC = z'B,$$
$$AB - BA = z''C.$$

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

Theorem

Assume that $\beta = -2$. *Then there exists* $C \in \mathrm{End}(V)$ *such that*

$$BC + CB = zA,$$
$$CA + AC = z'B,$$
$$AB + BA = z''C.$$

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

Theorem

Assume that $\beta \neq \pm 2$. *Pick* $0 \neq q \in \mathbb{F}$ *such that* $\beta = q^2 + q^{-2}$. *Then there exists* $C \in \mathrm{End}(V)$ *such that*

$$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = zA,$$

$$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = z'B,$$

$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = z''C.$$

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

We now make the above equations more attractive, by normalizing our TB tridiagonal pair.

Theorem

Assume that $\beta = 2$. Then for

$$\varrho = \varrho^* = 4, \qquad z = z' = z'' = 2\sqrt{-1}$$

we have

$$BC - CB = 2\sqrt{-1}A,$$
$$CA - AC = 2\sqrt{-1}B,$$
$$AB - BA = 2\sqrt{-1}C.$$

These are the defining relations for the Lie algebra $\mathfrak{sl}_2$ in the **Pauli presentation (1927)**.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

Theorem

Assume that $\beta = -2$. Then for

$$\varrho = \varrho^* = 4, \qquad z = z' = z'' = 2$$

we have

$$BC + CB = 2A,$$
$$CA + AC = 2B,$$
$$AB + BA = 2C.$$

These are the defining relations for the **anticommutator spin algebra**, due to **Arik and Kayserilioglu 2003**.

The $\mathbb{Z}_3$-symmetric Askey-Wilson relations, cont.

Theorem

Assume that $\beta \neq \pm 2$. Then for

$$\varrho = \varrho^* = 4 - \beta^2, \qquad z = z' = z'' = 1$$

we have

$$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = A,$$
$$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = B,$$
$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = C.$$

These are the defining relations for the **quantum group** $U_q(\mathfrak{so}_3)$. They first appeared in the work of **Santilli (1967)**.

The automorphism ρ

From now on, assume that A, B, C are normalized as above.

Our next goal is to display an automorphism ρ of $\mathrm{End}(V)$ that sends

$$A \mapsto B, \qquad B \mapsto C, \qquad C \mapsto A.$$

The automorphism ρ, cont.

Lemma

There exist invertible $W, W', W'' \in \mathrm{End}(V)$ *such that:*

(i) $AW = WA$ *and* $BW = WC$;

(ii) $BW' = W'B$ *and* $CW' = W'A$;

(iii) $CW'' = W''C$ *and* $AW'' = W''B$.

The automorphism ρ, cont.

Lemma

The following elements agree up to a nonzero scalar factor:

$$W'W, \qquad W''W', \qquad WW''.$$

Define $P = W'W$.

The automorphism ρ, cont.

Lemma

We have

$$AP = PB, \qquad BP = PC, \qquad CP = PA.$$

The automorphism ρ, cont.

Theorem

Let ρ denote the automorphism of $\mathrm{End}(V)$ that sends $X \mapsto P^{-1}XP$ for all $X \in \mathrm{End}(V)$. Then ρ sends

$$A \mapsto B, \qquad B \mapsto C, \qquad C \mapsto A.$$

Moreover $\rho^3 = 1$.

Definition

A **Leonard triple** on V is a 3-tuple of elements in $\mathrm{End}(V)$ such that for each map, there exists a basis for V with respect to which the matrix representing that map is diagonal and the matrix representing the other two maps are irreducible tridiagonal.

Theorem

The 3-tuple A, B, C is a Leonard triple on V.

Our next goal is to show that the Leonard triple A, B, C has a certain property, called **modular**.

Some antiautomorphisms

Define maps

$$\ddagger, \ddagger', \ddagger'' : \mathrm{End}(V) \to \mathrm{End}(V), \; X \mapsto T^{-1} X^{\dagger} T$$

where T is from the table below:

	$\ddagger$	$\ddagger'$	$\ddagger''$
T	W	$(W')^{-1}$	$WW'W$

Each map $\ddagger$, $\ddagger'$, $\ddagger''$ is an antiautomorphism of $\mathrm{End}(V)$.

Theorem

The antiautomorphisms $\ddagger$, $\ddagger'$, $\ddagger''$ act on A, B, C as follows:

- *$\ddagger$ fixes A and swaps B, C;*
- *$\ddagger'$ fixes B and swaps C, A;*
- *$\ddagger''$ fixes C and swaps A, B.*

The above theorem shows that the Leonard triple A, B, C is modular in the sense of Curtin.

Next we show that W, W', W'' satisfy some equations called the **braid relations**.

Definition

Elements $X, Y \in \mathrm{End}(V)$ are said to satisfy the **braid relation** whenever

$$XYX = YXY.$$

The braid relations

Theorem

Any two of W, W', W'' satisfy the braid relation.

We now bring in the **modular group** $\mathrm{PSL}_2(\mathbb{Z})$.

Recall that $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators r, s and relations $r^3 = 1$, $s^2 = 1$.

Our next goal is to display an action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{End}(V)$ as a group of automorphisms, that acts on A, B, C in an attractive manner.

An action of $\mathrm{PSL}_2(\mathbb{Z})$

To get our action of $\mathrm{PSL}_2(\mathbb{Z})$, we need an automorphism of $\mathrm{End}(V)$ that has order 3, and one that has order 2.

We already obtained an automorphism ρ of $\mathrm{End}(V)$ that has order 3.

Next we obtain an automorphism σ of $\mathrm{End}(V)$ that has order 2.

An action of $\mathrm{PSL}_2(\mathbb{Z})$, cont.

Definition

Define $T = WW'W$. Let σ denote the automorphism of $\mathrm{End}(V)$ that sends $X \mapsto TXT^{-1}$ for all $X \in \mathrm{End}(V)$.

Lemma

The automorphism σ swaps A, B and sends $C \mapsto C^{\dagger}$. Moreover $\sigma^2 = 1$.

An action of $\mathrm{PSL}_2(\mathbb{Z})$, cont.

Theorem

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathrm{End}(V)$ as a group of automorphisms, such that the generator r acts as ρ and the generator s acts as σ.

In this talk, we defined the TB tridiagonal pairs and systems.

We classified up to isomorphism the TB tridiagonal systems.

We showed how any TB tridiagonal pair satisfies the Askey-Wilson relations.

We put the Askey-Wilson relations in $\mathbb{Z}_3$-symmetric form.

We showed that each TB tridiagonal pair gives a modular Leonard triple.

We gave a related action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

Thank you for your attention!

THE END