Generalizing the Q-polynomial property to graphs that are not distance-regular

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There is a type of distance-regular graph, said to be Q-**polynomial**.

Many of the known distance-regular graphs have this property.

In this talk, we will generalize the Q-polynomial property to finite, undirected, connected graphs that are not necessarily distance-regular.

We will describe a family of examples associated with the **attenuated space** posets.

Let X denote a nonempty finite set.

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$.

Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{C}$.

The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

We call V the **standard module**.

Preliminaries, cont.

For all vertices $y \in X$, define a vector $\hat{y} \in V$ that has y-coordinate 1 and all other coordinates 0.

The vectors $\{\hat{y}\}_{y \in X}$ form a basis for V.

The graph Γ

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, adjacency relation $\mathcal{R}$, and path-length distance function ∂.

The weighted adjacency matrix

Definition

By a **weighted adjacency matrix of Γ**, we mean a matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ that has (y,z)-entry

$$A_{y,z} = \begin{cases} \neq 0, & \text{if } y, z \text{ are adjacent;} \\ 0, & \text{if } y, z \text{ are not adjacent} \end{cases} \qquad (y, z \in X).$$

Until further notice, we fix a weighted adjacency matrix A of Γ that is **diagonalizable over** $\mathbb{C}$.

The adjacency algebra

Let **M** denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.

We call **M** the **adjacency algebra** of Γ and A.

Let $\mathcal{D}+1$ denote the dimension of the vector space **M**.

The primitive idempotents

Since A is diagonalizable, the vector space $\mathbf{M}$ has a basis $\{E_i\}_{i=0}^{\mathcal{D}}$ such that

$$\sum_{i=0}^{\mathcal{D}} E_i = I,$$

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq \mathcal{D}).$$

We call $\{E_i\}_{i=0}^{\mathcal{D}}$ the **primitive idempotents of** A.

The scalars $\{\theta_i\}_{i=0}^{\mathcal{D}}$

Since $A \in \mathbf{M}$, there exist complex scalars $\{\theta_i\}_{i=0}^{\mathcal{D}}$ such that

$$A = \sum_{i=0}^{\mathcal{D}} \theta_i E_i.$$

The scalars $\{\theta_i\}_{i=0}^{\mathcal{D}}$ are mutually distinct since A generates $\mathbf{M}$.

The eigenspaces of A

Note that

$$V = \sum_{i=0}^{\mathcal{D}} E_i V \qquad \text{(direct sum)}.$$

For $0 \leq i \leq \mathcal{D}$ the subspace $E_i V$ is an eigenspace of A, and θ_i is the corresponding eigenvalue.

The base vertex x

Until further notice, fix a vertex $x \in X$, called the **base vertex**.

Define the integer $D = D(x)$ by

$$D = \max\{\partial(x, y) | y \in X\}.$$

We call D the **diameter of Γ with respect to** x.

We have $D \leq \mathcal{D}$, because the matrices $\{A^i\}_{i=0}^{D}$ are linearly independent.

The dual primitive idempotents

For $0 \leq i \leq D$ define a diagonal matrix $E_i^* = E_i^*(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

We call $\{E_i^*\}_{i=0}^D$ the **dual primitive idempotents of Γ with respect to** x.

We have

$$\sum_{i=0}^{D} E_i^* = I,$$

$$E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \leq i, j \leq D).$$

Consequently, the matrices $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $\mathbf{M}^* = \mathbf{M}^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$.

We call $\mathbf{M}^*$ the **dual adjacency algebra of Γ with respect to x.**

Definition

Let $\mathbf{T} = \mathbf{T}(x, A)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $\mathbf{M}$ and $\mathbf{M}^*$.

We call $\mathbf{T}$ the **subconsituent algebra of Γ with respect to x and A.**

Next we recall the subconstituents of Γ with respect to x.

For $0 \leq i \leq D$ we have

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in X,\ \partial(x, y) = i\}.$$

Moreover,

$$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(direct sum)}.$$

For $0 \leq i \leq D$ the subspace $E_i^* V$ is a common eigenspace for $\mathbf{M}^*$.

We call $E_i^* V$ the i^{th} **subconstituent of** Γ **with respect to** x.

The action of A on the subconstituents

By the triangle inequality, for adjacent vertices $y, z \in X$ the distances $\partial(x, y)$ and $\partial(x, z)$ differ by at most one.

Consequently

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

The dual adjacency matrix

Next we discuss the concept of a dual adjacency matrix.

Definition

A matrix $A^* \in \mathrm{Mat}_X(\mathbb{C})$ is called a **dual adjacency matrix of Γ with respect to x and the ordering** $\{E_i\}_{i=0}^{\mathcal{D}}$ whenever A^* generates $\mathbf{M}^*$ and

$$A^* E_i V \subseteq E_{i-1}V + E_i V + E_{i+1} V \qquad (0 \le i \le \mathcal{D}),$$

where $E_{-1} = 0$ and $E_{\mathcal{D}+1} = 0$.

The dual adjacency matrix, cont.

Definition

A matrix $A^* \in \mathrm{Mat}_X(\mathbb{C})$ is called a **dual adjacency matrix of Γ with respect to x and A** whenever A^* is a dual adjacency matrix with respect to x and some ordering of the primitive idempotents of A.

The Q-polynomial property

Next we introduce the (generalized) Q-polynomial property.

Definition

We say that A is **Q-polynomial with respect to** x whenever there exists a dual adjacency matrix A^* of Γ with respect to x and A.

The Attenuated Space poset $\mathcal{A}_q(N, M)$

For the rest of this talk, we illustrate the Q-polynomial property using an example.

This example is based on the **Attenuated Space poset** $\mathcal{A}_q(N, M)$.

This poset is defined on the next slides.

The definition of $\mathcal{A}_q(N, M)$

Let $\mathbb{F}_q$ denote a finite field with q elements.

Let N, M denote positive integers.

Let H denote a vector space over $\mathbb{F}_q$ that has dimension $N + M$.

Fix a subspace $h \subseteq H$ that has dimension M.

The definition of $\mathcal{A}_q(N, M)$, cont.

Let the set X consist of the subspaces of H that have zero intersection with h.

The set X, together with the containment relation, is a poset denoted by $\mathcal{A}_q(N, M)$ and called the **Attenuated Space poset**.

The poset $\mathcal{A}_q(N, M)$ is ranked with rank N; the rank of a vertex is equal to its dimension.

Next we define a graph Γ with vertex set X.

Vertices $y, z \in X$ are adjacent in Γ whenever one of y, z covers the other one.

In other words, the graph Γ is the **Hasse diagram** of the poset $\mathcal{A}_q(N, M)$.

Note that Γ is **bipartite**.

The rest of this talk is about Γ.

The base vertex x

Let $\mathbf{0}$ denote the zero subspace of H.

Recall the base vertex x of Γ.

For the rest of this talk, we choose $x = \mathbf{0}$.

Note that the diameter $D = D(x)$ is equal to N.

The q-adjacency matrix A

The following matrix A was introduced by S. Ghosh and M. Srinivasan in 2021.

Definition

Define a matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ that has (y, z)-entry

$$A_{y,z} = \begin{cases} 1 & \text{if } y \text{ covers } z; \\ q^{\dim y} & \text{if } z \text{ covers } y; \\ 0 & \text{if } y, z \text{ are not adjacent} \end{cases} \qquad y, z \in X.$$

The matrix A is a weighted adjacency matrix of Γ, called the q-**adjacency matrix**.

The q-adjacency matrix A, cont.

For the rest of the talk, our main goal is to show that:
the q-adjacency matrix A is Q-polynomial with respect to x.

Basic facts about $\mathcal{A}_q(N, M)$

Let us review some basic facts about $\mathcal{A}_q(N, M)$.

For an integer $n \geq 0$ define

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

We further define

$$[n]^!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]^!_q = 1$.

For $0 \leq i \leq n$ define

$$\binom{n}{i}_q = \frac{[n]^!_q}{[i]^!_q [n-i]^!_q}.$$

Basic facts about $\mathcal{A}_q(N, M)$, cont.

The following results are well known.

Lemma

Let $0 \leq i \leq N$ and let $y \in X$ have dimension i.

(i) *y covers exactly $[i]_q$ vertices;*

(ii) *y is covered by exactly $q^M[N-i]_q$ vertices.*

Lemma

For $0 \leq i \leq N$, the number of vertices in X that have dimension i is equal to $q^{Mi}\binom{N}{i}_q$.

The matrix A^*

Next, we define a matrix A^*.

Definition

Define a diagonal matrix $A^* \in \mathrm{Mat}_X(\mathbb{C})$ that has (y, y)-entry $q^{-\dim y}$ for $y \in X$.

Note that A^* is invertible.

We are going to show, that A^* **is a dual adjacency matrix of** Γ **with respect to** x **and** A.

Lemma

The matrix A^* has the following properties:

(i) *For $y \in X$,*

$$A^* \hat{y} = q^{-\dim y} \hat{y}.$$

(ii) *The eigenvalues of A^* are $\{q^{-i}\}_{i=0}^N$.*

(iii) *For $0 \leq i \leq N$,*
$E_i^* V$ is the eigenspace of A^* for the eigenvalue q^{-i}.

Some properties of A^*, cont.

Lemma (continued)

(iv) $A^* = \sum_{i=0}^{N} q^{-i} E_i^*$.

(v) *The algebra* $\mathbf{M}^*$ *is generated by* A^*.

(vi) *The algebra* $\mathbf{T}$ *is generated by* A, A^*.

Raising and Lowering matrices

As we investigate the algebra $\mathbf{T}$, it is useful to introduce three elements R, L, L'.

As we will explain, R is a **raising matrix** and L, L' are **lowering matrices**.

We comment on how L, L' differ.

The matrix L is natural from an **algebraic** point of view, and L' is natural from a **combinatorial** point of view.

Raising and Lowering matrices, cont.

We will define R, L' now.

We will define L a bit later.

The matrices R and L'

Definition

We define matrices R, L' in $\mathrm{Mat}_X(\mathbb{C})$ that have (y, z)-entries

$$R_{y,z} = \begin{cases} 1, & \text{if } y \text{ covers } z; \\ 0, & \text{if } y \text{ does not cover } z \end{cases}$$

$$L'_{y,z} = \begin{cases} 1, & \text{if } z \text{ covers } y; \\ 0, & \text{if } z \text{ does not cover } y \end{cases}$$

for $y, z \in X$. We call R (resp. L') the **raising matrix** (resp. **lowering matrix**) of $\mathcal{A}_q(N, M)$.

Lemma

For $z \in X$ we have

$$R\hat{z} = \sum_{y \text{ covers } z} \hat{y},$$

$$L'\hat{z} = \sum_{z \text{ covers } y} \hat{y}.$$

Lemma

We have

$$R = \sum_{i=0}^{N-1} E^*_{i+1} A E^*_i,$$

$$L' = \sum_{i=0}^{N-1} q^{-i} E^*_i A E^*_{i+1}.$$

Moreover $R, L' \in \mathbf{T}$.

The matrices R, L', A^* generate $\mathbf{T}$

Lemma

The matrices R, L', A^* satisfy the following:

(i) $L' = R^t$, *where t denotes transpose;*

(ii) $A = R + (A^*)^{-1}L'$;

(iii) *the algebra* $\mathbf{T}$ *is generated by* R, L', A^*;

(iv) $\mathbf{T}$ *is closed under transpose and complex-conjugation;*

(v) $\mathbf{T}$ *is semisimple.*

How R, L', A^* are related

Lemma (Wen Liu 2016)

The matrices R, L', A^* are related as follows:

$$RA^* = qA^*R, \qquad L'A^* = q^{-1}A^*L',$$
$$(L')^2R - (q+1)L'RL' + qR(L')^2 = -(q+1)q^{N+M}L'A^*,$$
$$L'R^2 - (q+1)RL'R + qR^2L' = -(q+1)q^{N+M}A^*R.$$

The above relations express the fact that the poset $\mathcal{A}_q(N, M)$ is **uniform** in the sense of [Ter 1990].

The $\mathbf{T}$-modules

Next, we describe the $\mathbf{T}$-modules.

Recall the standard module V.

By a **$\mathbf{T}$-module** we mean a subspace $W \subseteq V$ such that $\mathbf{T}W \subseteq W$.

A $\mathbf{T}$-module W is said to be **irreducible** whenever $W \neq 0$ and W does not contain a $\mathbf{T}$-module besides 0 and W.

Because $\mathbf{T}$ is semisimple, the standard module V is a direct sum of irreducible $\mathbf{T}$-modules.

Next, we describe an irreducible **T**-module W.

By the **endpoint** of W we mean $\min\{i|0 \le i \le N,\ E_i^* W \neq 0\}$.

By the **diameter** of W we mean $\big|\{i|0 \le i \le N,\ E_i^* W \neq 0\}\big| - 1$.

The irreducible **T**-modules, cont.

Using the theory of uniform posets (Ter 1990) we obtain the following result.

Lemma (Ter 1990)

For $0 \leq r, d \leq N$ the following are equivalent:

(i) *there exists an irreducible* **T***-module with endpoint r and diameter d;*

(ii) $N - 2r \leq d \leq N - r$ *and* $d \leq N + M - 2r$.

We mention one significance of the endpoint and diameter.

Lemma

Let W and W' denote irreducible $\mathbf{T}$-modules, with endpoints r, r' and diameters d, d' respectively. Then the following are equivalent:

(i) *W and W' are isomorphic;*

(ii) *$r = r'$ and $d = d'$.*

The irreducible **T**-modules, cont.

let Ψ denote the set of isomorphism classes of irreducible **T**-modules. By the previous lemma, we view

$$\Psi = \{(r, d) \mid 0 \le r, d \le N, \quad N - 2r \le d \le N - r, \\ d \le N + M - 2r\}.$$

The irreducible **T**-modules, cont.

We bring in some notation.

Definition

For $(r, d) \in \Psi$ define

$$\xi_i'(r, d) = \frac{q^{N+M-r-d}(q^i - 1)(q^{d+1-i} - 1)}{(q-1)^2} \qquad (1 \leq i \leq d).$$

The irreducible **T**-modules, cont.

Using the theory of uniform posets (Ter 1990) we get the following result.

Lemma (Ter 1990)

Let W denote an irreducible $\mathbf{T}$-module, with endpoint r and diameter d. There exists a basis $\{w_i\}_{i=0}^{d}$ of W such that

(i) $w_i \in E^*_{r+i}V \quad (0 \le i \le d)$;

(ii) $Rw_i = w_{i+1} \quad (0 \le i \le d-1), \qquad Rw_d = 0$;

(iii) $L'w_i = \xi'_i(r,d)w_{i-1} \quad (1 \le i \le d), \qquad L'w_0 = 0.$

The q-lowering matrix L

Next, we adjust the lowering matrix L' to get a matrix L called the q-lowering matrix.

Definition

We define a matrix $L \in \mathrm{Mat}_X(\mathbb{C})$ that has (y, z)-entry

$$L_{y,z} = \begin{cases} q^{\dim y}, & \text{if } z \text{ covers } y; \\ 0, & \text{if } z \text{ does not cover } y \end{cases} \qquad (y, z \in X).$$

We call L the q-**lowering matrix** for $\mathcal{A}_q(N, M)$.

The q-lowering matrix L, cont.

Lemma

For $z \in X$ we have

$$L\hat{z} = \sum_{z \text{ covers } y} q^{\dim y}\hat{y}.$$

The q-lowering matrix L, cont.

Lemma

We have $L' = A^* L$ *and* $A = R + L$.

Lemma

The algebra $\mathbf{T}$ *is generated by* R, L, A^*.

How R, L, A^* are related

Lemma

The matrices R, L, A^* are related as follows:

$$RA^* = qA^*R, \qquad LA^* = q^{-1}A^*L,$$
$$L^2R - q(q+1)LRL + q^3RL^2 = -q^{N+M}(q+1)L,$$
$$LR^2 - q(q+1)RLR + q^3R^2L = -q^{N+M}(q+1)R.$$

The previous two equations are the **Down-Up Relations** due to **Benkart** and **Roby** (1998).

Next, we describe how L acts on an irreducible T-module.

Definition

For $(r, d) \in \Psi$ define

$$\xi_i(r,d) = \frac{q^{N+M-d}(q^i - 1)(q^d - q^{i-1})}{(q-1)^2} \qquad (1 \le i \le d).$$

Note that $\xi_i(r,d) = q^{r+i-1}\xi'_i(r,d)$ for $1 \le i \le d$.

Lemma

Let W denote an irreducible T-module, with endpoint r and diameter d. Recall the basis $\{w_i\}_{i=0}^d$ of W. Then

$$Lw_i = \xi_i(r,d)w_{i-1} \qquad (1 \le i \le d), \qquad Lw_0 = 0.$$

The matrices A and A^*

We have been discussing R, L, A^*.

Let us return our attention to A, A^*.

Our next goals are:

- determine how A, A^* are related;
- show that A is diagonalizable;
- find the eigenvalues of A;
- describe how A^* acts on the eigenspaces of A.

How A and A^* are related

We now describe how A, A^* are related.

For notational convenience, define

$$\beta = q + q^{-1}.$$

Lemma

The matrices A and A^* satisfy

$$A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 = q^{N+M-2}(q+1)^2(AA^* - A^*A),$$

$$(A^*)^2A - \beta A^*AA^* + A(A^*)^2 = 0.$$

Next, we show that A is diagonalizable, and we find its eigenvalues.

To do this, we consider the action of A on each irreducible **T**-module.

The eigenvalues of A

Lemma

Let W denote an irreducible $\mathbf{T}$-module, with endpoint r and diameter d. Consider the matrix that represents A with respect to the basis $\{w_i\}_{i=0}^d$ of W. This matrix is tridiagonal with entries

$$\begin{pmatrix} 0 & \xi_1 & & & & \mathbf{0} \\ 1 & 0 & \xi_2 & & & \\ & 1 & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \xi_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix},$$

where $\xi_i = \xi_i(r, d)$.

The matrix on the previous slide is well known in the theory of Leonard pairs.

Shortly we will describe its eigenvalues.

We bring in some notation.

Definition

Define the set

$$[N] = \{i \mid 2i \in \mathbb{Z}, \quad 0 \leq i \leq N\}$$
$$= \{0, 1/2, 1, 3/2, \ldots, N\}.$$

The cardinality of $[N]$ is $2N + 1$.

The eigenvalues of A, cont.

The scalar q is real and $q \geq 2$; let $q^{1/2}$ denote the positive square root of q.

Definition

For $i \in [N]$ define

$$\theta_i = \frac{q^{N-i} - q^i}{q-1} \, q^{M/2}.$$

Lemma

The scalars $\{\theta_i\}_{i \in [N]}$ are mutually distinct.

The eigenvalues of A, cont.

Definition

Consider a linear map on a finite-dimensional vector space.

This map is called **multiplicity-free** whenever the map is diagonalizable, and each eigenspace has dimension one.

The eigenvalues of A, cont.

Lemma

Let W denote an irreducible **T***-module. Let d denote the diameter of W, and define $t = (N - d)/2$. Then the action of A on W is multiplicity-free, with eigenvalues $\{\theta_{t+i}\}_{i=0}^{d}$.*

The eigenvalues of A, cont.

Lemma

The matrix A is diagonalizable, with eigenvalues $\{\theta_i\}_{i \in [N]}$.

Corollary

The dimension of $\mathbf{M}$ is $2N + 1$.

The action of A^* on the eigenspaces of A

Next, we describe how A^* acts on the eigenspaces of A.

Lemma

For distinct $i, j \in [N]$,

$$E_i A^* E_j \neq 0 \quad \text{if and only if} \quad |i - j| = 1.$$

Proof.

Recall that

$$A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3$$
$$= q^{N+M-2}(q+1)^2(AA^* - A^*A).$$

In this equation, multiply each term on the left by E_i and the right by E_j. Simplify the result using $E_iA = \theta_i E_i$ and $AE_j = \theta_j E_j$. □

We can now easily show that A^* is a dual adjacency matrix.

Lemma

The matrix A^* is a dual adjacency matrix with respect to x and the following orderings of the primitive idempotents:

(i) $E_0 < E_1 < E_2 < \cdots < E_N < E_{1/2} < E_{3/2} < \cdots < E_{N-1/2}$;

(ii) $E_{1/2} < E_{3/2} < \cdots < E_{N-1/2} < E_0 < E_1 < E_2 < \cdots < E_N$.

The matrix A^* is a dual adjacency matrix, cont.

Corollary

The matrix A^* is a dual adjacency matrix of Γ with respect to x and A.

The main result

We now state our main result.

Theorem

The matrix A is Q-polynomial with respect to x.

Summary

In this talk, we first generalized the Q-**polynomial property** to graphs that are not necessarily distance-regular.

We defined a graph Γ using the **Attenuated Space poset** $\mathcal{A}_q(N, M)$.

We considered Γ from the point of view of the **base vertex** $x = \mathbf{0}$.

We defined the q-**adjacency matrix** A and a **diagonal matrix** A^*.

We showed that A^* is a **dual adjacency matrix** of Γ with respect to x and A.

Finally we showed that A is Q-**polynomial with respect to** x.

THANK YOU FOR YOUR ATTENTION!